

## Extra exercises on proof strategies

### Explanation – read this first

This handout contains several additional exercises to practice using proof strategies. Read the corresponding chapter first and familiarize yourself with the basic proof strategies, notably those for conjunction, disjunction, and implication.

In addition to these proof strategies, you may use the following definitions:

- $A \cap B$  is defined as  $\{x : x \in A \wedge x \in B\}$ . In particular,  $x \in A \cap B$  if and only if  $x \in A \wedge x \in B$
- $A \cup B$  is defined as  $\{x : x \in A \vee x \in B\}$ . In particular,  $x \in A \cup B$  if and only if  $x \in A \vee x \in B$
- $A \setminus B$  is defined as  $\{x : x \in A \wedge x \notin B\}$ . In particular,  $x \in A \setminus B$  if and only if  $x \in A \wedge \neg(x \in B)$
- $A \subseteq B$  is defined as  $\forall x, x \in A \Rightarrow x \in B$

There are two types of questions: initially, you will need to analyse a proof to identify the strategies that have been used; later, you will need to write such proofs by hand. Take the time to write out as many details as possible for the first exercises. Once you get the hang of things, it is fine to leave certain obvious steps implicit (such as repeated implication introduction, introducing universally quantified variables, etc.).

Finally, I strongly suggest printing this handout—it makes it much easier to do the first two exercises on the exercise sheet itself.

### Exercises

1. Consider the following proof establishing that for all sets  $A$  and  $B$ , the statement  $A \cap B \subseteq B \cap A$  holds. Read the proof carefully and try to identify the proof strategies used. Draw boxes around the text and label them with the proof strategy applied in each step.

**Theorem 1.**  $\forall A \forall B \quad A \cap B \subseteq B \cap A$

*Proof.* Let  $A$  and  $B$  be arbitrary sets.

We want to show that  $A \cap B \subseteq B \cap A$ .

By definition of subset operator,  $\subseteq$ , this amounts to proving

$$\forall x, x \in A \cap B \Rightarrow x \in B \cap A.$$

Let  $x$  be arbitrary.

Assume  $x \in A \cap B$ .

As  $x \in A \cap B$  holds by assumption, we know that  $x \in B$

As  $x \in A \cap B$  holds by assumption, we know that  $x \in A$

Therefore  $x \in B \cap A$

Therefore,  $A \cap B \subseteq B \cap A$

□

**Solution:**

Let  $A$  and  $B$  be arbitrary sets.

$\forall$ -I

We want to show that  $A \cap B \subseteq B \cap A$ .

By definition of subset operator,  $\subseteq$ , this amounts to proving

$$\forall x, x \in A \cap B \Rightarrow x \in B \cap A.$$

Let  $x$  be arbitrary.

$\forall$ -I

Assume  $x \in A \cap B$ .

$\Rightarrow$ -I

$\wedge$ -I

As  $x \in A \cap B$  holds by assumption, we know that  $x \in B$       $\wedge$ -E

As  $x \in A \cap B$  holds by assumption, we know that  $x \in A$       $\wedge$ -E

As  $x \in B$  and  $x \in A$ , we conclude  $x \in B \cap A$

Therefore,  $A \cap B \subseteq B \cap A$

2. Consider the following proof establishing that for all sets  $A$  and  $B$ , the statement  $A \cup B \subseteq B \cup A$  holds. Read the proof carefully and try to identify the proof strategies used. Draw boxes around the text and label them with the proof strategy applied in each step.

**Theorem 2.**  $\forall A \forall B \quad A \cup B \subseteq B \cup A$

*Proof.* Let  $A$  and  $B$  be arbitrary sets.

We want to show that  $A \cup B \subseteq B \cup A$ .

By definition of subset operator,  $\subseteq$ , this amounts to proving

$$\forall x, x \in A \cup B \Rightarrow x \in B \cup A.$$

Let  $x$  be arbitrary.

Assume  $x \in A \cup B$ .

Thus,  $x \in A \vee x \in B$  is true, by definition of set union.

First, assume  $x \in A$ . Then  $x \in B \vee A$  must be true.

Hence, by definition of set union, we know  $x \in B \cup A$ . Next assume  $x \in B$ . Then  $x \in B \vee A$  must be true.

Then, once again,  $x \in B \cup A$  must be true by definition of set union.

Therefore  $x \in B \cup A$  (regardless of whichever  $x \in A$  is true or  $x \in B$  is true).

Therefore,  $x \in A \cup B \Rightarrow x \in B \cup A$ .

Therefore,  $A \cup B \subseteq B \cup A$  as required. □

**Solution:**

Let  $A$  and  $B$  be arbitrary sets.  $\forall$ -I

We want to show that  $A \cup B \subseteq B \cup A$ . By definition of subset operator,  $\subseteq$ , this amounts to proving

$$\forall x, x \in A \cup B \Rightarrow x \in B \cup A.$$

Let  $x$  be arbitrary.  $\forall$ -I

Assume  $x \in A \cup B$ .  $\Rightarrow$ -I

Thus,  $x \in A \vee x \in B$  is true, by definition of set union.

First, assume  $x \in A$ . Then  $x \in B \vee A$  must be true.  $\vee$ -I  
Hence, by definition of set union, we know  $x \in B \cup A$ .

Next assume  $x \in B$ . Then  $x \in B \vee A$  must be true.  
Then, once again,  $x \in B \cup A$  by definition of set union.

Therefore  $x \in B \cup A$  (regardless of whichever  $x \in A$  is true or  $x \in B$  is true).

Therefore,  $x \in A \cup B \Rightarrow x \in B \cup A$ .

Therefore,  $A \cup B \subseteq B \cup A$  as required.

3. Use the proof strategies presented in class to give a proof of the following theorems.

Explicitly mark any proof strategies used. Be as precise as possible about any other steps in your reasoning.

(a)  $\forall A \forall B \forall C \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C$

**Solution:**

Let  $A$ ,  $B$ , and  $C$  be arbitrary sets and let  $x$  be arbitrary.  $\forall$ -I

We need to show:  $x \in A \cap (B \cap C) \Rightarrow x \in (A \cap B) \cap C$ .

Assume that  $x \in A \cap (B \cap C)$   $\Rightarrow$ -I

By definition of set intersection, we know  $x \in A \wedge (x \in B \wedge x \in C)$ .

$\wedge$ -I

$\wedge$ -I

$\wedge$ -E

Using our assumption, we know that  $x \in A$ .

$\wedge$ -E

Using our assumption, we know that  $x \in B \wedge x \in C$ .

$\wedge$ -E

Hence  $x \in B$ .

From the above we conclude that  $x \in A \wedge x \in B$ .

$\wedge$ -E

Using our assumption, we know that  $x \in B \wedge x \in C$ .

$\wedge$ -E

Hence  $x \in C$ .

Hence we conclude that  $(x \in A \wedge x \in B) \wedge x \in C$ . By definition of set intersection, we conclude that  $x \in (A \cap B) \cap C$ .

Therefore that  $x \in A \cap (B \cap C) \Rightarrow x \in (A \cap B) \cap C$  as required.

(b)  $\forall A B \quad A \cup (A \cap B) \subseteq A$

**Solution:**

Let A and B be arbitrary sets and let x be arbitrary.

$\forall$ -I

We need to show:  $x \in A \cup (A \cap B) \Rightarrow x \in A$ .

$\Rightarrow$ -I

Assume that  $x \in A \cup (A \cap B)$ .

By definition of intersection and union, we know that  $x \in A \vee (x \in A \wedge x \in B)$ . We now distinguish two possible cases:

$\vee$ -E

First, if  $x \in A$ , clearly  $x \in A$  as required.

Next, assume that  $x \in A \wedge x \in B$ .

$\wedge$ -E

Then  $x \in A$  as required.

Therefore,  $x \in A$  (regardless of whichever of  $x \in A$  or  $(x \in A \wedge x \in B)$  is true).

Hence, we conclude that  $x \in A \cup (A \cap B) \Rightarrow x \in A$  as required.

Hence, for all sets A and B, we have shown  $A \cup (A \cap B) \subseteq A$

(c)  $\forall A \quad A \cup \emptyset \subseteq A$

(Hint: you will need to use the falsity elimination rule from the lectures.)

**Solution:**

$\forall$ -I

Let  $A$  and  $x$  be arbitrary. We need to show that  $x \in A \cup \emptyset \Rightarrow x \in A$ .

$\Rightarrow$ -I

Assume that  $x \in A \cup \emptyset$ . By the definition of set union, we know that  $x \in A \vee x \in \emptyset$ . We now distinguish two possible cases and establish  $x \in A$  holds in both of them.

Firstly, suppose  $x \in A$ . Then clearly  $x \in A$ .

$\perp$ -E

Next, suppose  $x \in \emptyset$ . Then we have reached a contradiction, as the empty set has no elements.

Hence we can conclude that  $x \in A$ .

Therefore,  $x \in A$  (regardless of whichever of  $x \in A$  or  $x \in \emptyset$  is true). Hence, we conclude that  $x \in A \cup \emptyset \Rightarrow x \in A$ .

Therefore, we have established that for all sets  $A$ , we can show  $A \cup \emptyset \subseteq A$ .

(d)  $\forall A \forall B \forall C \quad (A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$

**Solution:**

$\forall$ -I

Let A, B and C be arbitrary sets. Let x be arbitrary.

We need to prove that  $x \in (A \cup B) \setminus C \Rightarrow x \in (A \setminus C) \cup (B \setminus C)$

Assume that  $x \in (A \cup B) \setminus C$ .

By definition of set difference and union, we know that  $x \in (A \cup B) \wedge x \notin C$ .

Hence we know that  $x \in A \cup B$ .

$\wedge$ -E

And we know that  $x \notin C$ .

$\wedge$ -E

We now show that  $x \in (A \setminus C) \cup (B \setminus C)$  by distinguishing two possible cases:

$\vee$ -E

Firstly, assume that  $x \in A$ . By our assumption, we know that  $x \notin C$ .

$\wedge$ -I

Hence we conclude that  $x \in A \wedge x \notin C$ .

From this we deduce that,  $(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ .

$\vee$ -I

Next, assume that  $x \in B$ . By our assumption, we know that  $x \notin C$ .

$\wedge$ -I

Hence we conclude that  $x \in B \wedge x \notin C$ .

And furthermore,  $(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ .

$\vee$ -I

Hence we know that  $(x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ , regardless of whether  $x \in A$  or  $x \in B$ .

Using the definitions of set union and difference, we conclude  $x \in (A \setminus C) \cup (B \setminus C)$ .

Therefore  $x \in (A \cup B) \setminus C \Rightarrow x \in (A \setminus C) \cup (B \setminus C)$  as required.

(e)  $\forall A \forall B \forall C \quad A \subseteq C \vee B \subseteq C \Rightarrow A \cap B \subseteq C$

**Solution:**



Let  $A, B, C$ , and  $x$  be arbitrary. We need to show that  
 $A \subseteq C \vee B \subseteq C \Rightarrow A \cap B \subseteq C$ .

$\forall$ -I

Assume that  $A \subseteq C \vee B \subseteq C$ .

$\Rightarrow$ -I

We now need to show that  $A \cap B \subseteq C$ .

Unfolding the definition of subsets, this amounts to proving:

$\forall x \in A \cap B \Rightarrow x \in C$ .

Let  $x$  be arbitrary. We need to show that  $x \in A \cap B \Rightarrow x \in C$ .

$\forall$ -I

Assume that  $x \in A \cap B$ . By definition of set intersection, we know:  
 $x \in A \wedge x \in B$ . We now prove that  $x \in C$  by distinguishing two cases:

$\Rightarrow$ -I

Firstly, suppose  $A \subseteq C$ .

$\forall$ -E

From our assumption  $x \in A \wedge x \in B$ , we know  
that  $x \in A$ .

$\wedge$ -E

Hence we conclude that  $x \in C$ .

$\forall$ -E and  $\Rightarrow$ -E

Next, suppose  $B \subseteq C$ .

From our assumption  $x \in A \wedge x \in B$ , we know  
that  $x \in B$ .

$\wedge$ -E

Hence we conclude that  $x \in C$ .

$\forall$ -E and  $\Rightarrow$ -E

Hence  $x \in C$ , regardless of whichever  $A \subseteq C$  or  $B \subseteq C$  holds.

Hence  $x \in A \cap B \Rightarrow x \in C$ .

Hence  $A \cap B \subseteq C$ .

Hence  $A \subseteq C \vee B \subseteq C \Rightarrow A \cap B \subseteq C$  as required.

4. Give a proof of the following theorems. Use the proof strategies from class to structure your proof, but you no longer need to identify each strategy being used explicitly.

(a) If the sets  $A$  and  $B$  are disjoint,  $A \setminus B = A$ .

**Solution:** Let  $A$  and  $B$  be arbitrary sets.

Assume that  $A$  and  $B$  are disjoint, that is  $A \cap B = \emptyset$ .

We need to prove that  $A \setminus B = A$ .

To do so, we prove  $A \setminus B \subseteq A$  and  $A \subseteq A \setminus B$  separately.

- First, we prove  $A \setminus B \subseteq A$ .

Let  $x \in A \setminus B$  be arbitrary. By definition of set difference, we know:

$$x \in A \wedge x \notin B.$$

We need to show that  $x \in A$ , but this follows from our assumption.

- Next, we prove  $A \subseteq A \setminus B$ .

Let  $x \in A$  be arbitrary. We need to show that  $x \in A \setminus B$ , that is,  $x \in A \wedge x \notin B$ .

From our assumption, we already know that  $x \in A$ .

We still need to prove  $x \notin B$ .

To do, assume that  $x \in B$ . Then, by our assumption that  $x \in A$ , we have  $x \in A \cap B$  – but this contradicts our assumption that  $A$  and  $B$  are disjoint. (negation introduction)

Hence we conclude that  $x \notin B$  as required.

Hence  $A \setminus B = A$  as required.

(b) For all sets  $A$  and  $B$ , if  $A \subseteq B$ , then  $A \cup B = B$ .

**Solution:** Let  $A$  and  $B$  be arbitrary sets. Assume that  $A \subseteq B$ .

We need to prove that  $A \cup B = B$ .

To do so, we prove both  $A \cup B \subseteq B$  and  $B \subseteq A \cup B$ .

- Let  $x \in A \cup B$ . We need to prove that  $x \in B$ .

We distinguish two possible cases:

- Firstly, suppose  $x \in A$ . From our assumption that  $A \subseteq B$ , we conclude  $x \in B$  as required.
- Secondly, suppose  $x \in B$ . Then clearly,  $x \in B$  as required.

Hence we conclude that  $A \cup B \subseteq B$ .

- Let  $x \in B$ . By definition of set union, we know that  $x \in A \cup B$ .

Therefore,  $A \cup B = B$ , provided  $A \subseteq B$ .

(c) For all sets  $A$  and  $B$ , if  $A \cap B = A$  then  $A \subseteq B$ .

**Solution:** Let  $A$  and  $B$  be arbitrary sets. Assume that  $A \cap B = A$ .

To prove that  $A \subseteq B$ , we assume  $x \in A$  and need to show  $x \in B$ .

From our assumption, we know that  $A \subseteq A \cap B$ .

Hence we know that  $x \in A \cap B$  – and therefore in particular that  $x \in B$  as required.

(d) For all sets  $A$ ,  $B$  and  $C$ , we have that  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .

**Solution:** Let  $A$ ,  $B$ , and  $C$  be arbitrary sets. Let  $x \in (A \cap B) \cup (A \cap C)$  be arbitrary.

We need to prove that  $x \in A \cap (B \cup C)$ .

We distinguish two possible cases:

- Suppose  $x \in A \cap B$ , by definition of intersection, we know  $x \in A$  and  $x \in B$ .  
Then clearly also  $x \in A$  and  $x \in (B \cup C)$ , hence we are done.
- Suppose  $x \in A \cap C$ , by definition of intersection, we know  $x \in A$  and  $x \in C$ .  
Then clearly also  $x \in A$  and  $x \in (B \cup C)$ , hence we are done.

Hence we conclude that  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .