Extra exercises on proof strategies

Explanation – read this first

This handout contains several additional exercises to practice using proof strategies. Read the corresponding chapter first and familiarize yourself with the basic proof strategies, notably those for conjunction, disjunction, and implication.

In addition to these proof strategies, you may use the following definitions:

- $A \cap B$ is defined as $\{ x : x \in A \wedge x \in B\}$. In particular, $x \in A \cap B$ if and only if $x \in A \wedge x \in B$
- $A \cup B$ is defined as $\{ x : x \in A \vee x \in B\}$. In particular, $x \in A \vee B$ if and only if $x \in A \vee x \in B$
- $A \setminus B$ is defined as $\{x : x \in A \wedge x \notin B\}$. In particular, $x \in A \setminus B$ if and only if $x \in A \wedge \neg (x \in B)$
- $A \subseteq B$ is defined as $\forall x,\ x \in A \Rightarrow x \in B$

There are two types of questions: initially, you will need to analyse a proof to identify the strategies that have been used; later, you will need to write such proofs by hand. Take the time to write out as many details as possible for the first exercises. Once you get the hang of things, it is fine to leave certain obvious steps implicit (such as repeated implication introduction, introducing universally quantified variables, etc.).

Finally, I strongly suggest printing this handout—it makes it much easier to do the first two exercises on the exercise sheet itself.

Exercises

1. Consider the following proof establishing that for all sets A and B, the statement $A \cap B \subseteq B \cap A$ holds. Read the proof carefully and try to identify the proof strategies used. Draw boxes around the text and label them with the proof strategy applied in each step.

   Theorem 1. $\forall A \forall B \quad A \cap B \subseteq B \cap A$

   Proof. Let A and B be arbitrary sets.

   We want to show that $A \cap B \subseteq B \cap A$.

   By definition of subset operator, $\subseteq$, this amounts to proving

   $\forall x, x \in A \cap B \Rightarrow x \in B \cap A$.

   Let x be arbitary.

   Assume $x \in A \cap B$.

   As $x \in A \cap B$ holds by assumption, we know that $x \in B$

   As $x \in A \cap B$ holds by assumption, we know that $x \in A$

   Therefore $x \in B \cap A$

   Therefore, $A \cap B \subseteq B \cap A$

   □

2. Consider the following proof establishing that for all sets A and B, the statement $A \cup B \subseteq B \cup A$ holds. Read the proof carefully and try to identify the proof strategies used. Draw boxes around the text and label them with the proof strategy applied in each step.

Theorem 2. $\forall A\,\forall B \quad A \cup B \subseteq B \cup A$

Proof. Let A and B be arbitrary sets.

We want to show that A $\cup$ B $\subseteq$ B $\cup$ A.

By definition of subset operator, $\subseteq$, this amounts to proving

$\forall$ x, x $\in$ A $\cup$ B $\Rightarrow$ x $\in$ B $\cup$ A.

Let x be arbitary.

Assume x $\in$ A $\cup$ B.

Thus, x $\in$ A $\vee$ x $\in$ B is true, by definition of set union.

First, assume x $\in$ A. Then x $\in$ B $\vee$ A must be true.

Hence, by definition of set union, we know x $\in$ B $\cup$ A. Next assume x $\in$ B. Then x $\in$ B $\vee$ A must be true.

Then, once again, x $\in$ B $\cup$ A must be true by definition of set union.

Therefore x $\in$ B $\cup$ A (regardless of whichever x $\in$ A is true or x $\in$ B is true).

Therefore, x $\in$ B $\cup$ A $\Rightarrow$ x $\in$ B $\cup$ A.

Therefore, A $\cup$ B $\subseteq$ B $\cup$ A as required.

$\square$

3. Use the proof strategies presented in class to give a proof of the following theorems.

   Explicitly mark any proof strategies used. Be as precise as possible about any other steps in your reasoning.

   (a) $\forall A\,\forall B\,\forall C \quad A \cap (B \cap C) \subseteq (A \cap B) \cap C$

   (b) $\forall A\, B \quad A \cup (A \cap B) \subseteq A$

   (c) $\forall A \quad A \cup \emptyset \subseteq A$

   (Hint: you will need to use the falsity elimination rule from the lectures.)

   (d) $\forall A\,\forall B\,\forall C \quad (A \cup B)\backslash C \subseteq (A\backslash C) \cup (B\backslash C)$

   (e) $\forall A\,\forall B\,\forall C \quad A \subseteq C \vee B \subseteq C \Rightarrow A \cap B \subseteq C$

4. Give a proof of the following theorems. Use the proof strategies from class to structure your proof, but you no longer need to identify each strategy being used explicitly.

   (a) If the sets A and B are disjoint, $A\backslash B = A$.

   (b) For all sets A and B, if A $\subseteq$ B, then A $\cup$ B = B.

   (c) For all sets A and B, if A $\cap$ B = A then A $\subseteq$ B.

   (d) For all sets A, B and C, we have that (A $\cap$ B) $\cup$ (A $\cap$ C) $\subseteq$ A $\cap$ (B $\cup$ C).