$n \times n$ matrices

The system of m linear equations in n variables $x_1, x_2, \ldots, x_n$

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

can be written as a matrix equation by $Ax = b$, or in full

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$n \times n$ matrices

The matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

has m rows and n columns, and is called an $m \times n$ matrix.

Special matrices

A square matrix (for which $m = n$) is called a diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero. If all elements a_{ii} are one, then the matrix is called an identity matrix, denoted with I_m (depending on the context, the subscript m may be left out).

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

If all matrix entries are zero, then the matrix is called a zero matrix or null matrix, denoted with 0.

Matrix addition

For two matrices A and B, we have $A + B = C$, with $c_{ij} = a_{ij} + b_{ij}$:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{bmatrix}$$

Q: what are the conditions for the dimensions of the matrices A and B?

Matrix multiplication

Multiplying a matrix with a scalar is defined as follows: $cA = B$, with $b_{ij} = ca_{ij}$. For example,

$$2\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}2 & 4 & 6\\8 & 10 & 12\\14 & 16 & 18\end{bmatrix}$$

Matrix multiplication

Multiplying two matrices is a bit more involved.
We have $AB = C$ with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For example,

$$\begin{bmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{bmatrix}$$

Q: what are the conditions for the dimensions of the matrices A and B? What are the dimensions of C?

Properties of matrix multiplication

Matrix multiplication is associative and distributive over addition:

$$
\begin{aligned}
(AB)C &= A(BC) \\
A(B+C) &= AB + AC \\
(A+B)C &= AC + BC
\end{aligned}
$$

However, matrix multiplication is not commutative:
in general, $AB \neq BA$.

Also: if $AB = AC$, it doesn't necessarily follow that $B = C$ (even if A is not the zero matrix).

Zero and identity matrix

The zero matrix 0 has the property that if you add it to another matrix A, you get precisely A again.

$$A + 0 = 0 + A = A$$

The identity matrix I has the property that if you multiply it with another matrix A, you get precisely A again.

$$AI = IA = A$$

Matrix multiplication as a linear transformation: 2D

The matrix multiplication of a 2×2 square matrix and a 2×1 matrix gives a new 2×1 matrix, e.g.:

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

We can interpret a 2×1 matrix as a vector; the 2×2 matrix transforms any vector into another vector.

More later ...

Transposed matrices

The transpose A^T of an $m \times n$ matrix A is an $n \times m$ matrix that is obtained by interchanging the rows and columns of A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Transposed matrices

for example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

For the transpose of the product of two matrices we have

$$(AB)^T = B^T A^T$$

The dot product revisited

If we regard (column) vectors as matrices, we see that the inproduct of two vectors can be written as $u \cdot v = u^T v$:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32$$

(A 1×1 matrix is simply a number, and the brackets are omitted.)

Inverse matrices

The inverse of a matrix A is a matrix A^{-1} such that $AA^{-1} = I$.

Only square matrices possibly have an inverse.

Note that the inverse of A^{-1} is A, so we have $AA^{-1} = A^{-1}A = I$

Gaussian elimination

Matrices are a convenient way of representing systems of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

If such a system has a unique solution, it can be solved with Gaussian elimination.

Gaussian elimination

Permitted operations in Gaussian elimination are

- interchanging two rows.
- multiplying a row with a (non-zero) constant.
- adding a multiple of another row to a row.

Gaussian elimination

Matrices are not necessary for Gaussian elimination, but very convenient, especially augmented matrices. The augmented matrix corresponding to the system of equations on the previous slides is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gaussian elimination: example

Suppose we want to solve the following system:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

Q: what is the geometric interpretation of this system? And what is the interpretation of its solution?

Gaussian elimination: example

Applying the rules in a clever order, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 1 & 1 & 9 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & 1 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Gaussian elimination: example

The interpretation of the last augmented matrix $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right]$

is the very convenient system of linear equations $\begin{aligned} x &= 3 \\ y &= 4 \\ z &= 5 \end{aligned}$

In other words, the point $(3, 4, 5)$ satisfies all three equations.

Gaussian elimination: geometric interpretation

We started with three equations, which are implicit representations of planes:

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

We ended with three other equations, which can also be interpreted as planes:

$$\begin{aligned} x &= 3 \\ y &= 4 \\ z &= 5 \end{aligned}$$

The steps in Gaussian elimination preserve the location of the solution.

Gaussian elimination: possible outcomes in 3D

Since any linear equation in three variables is a plane in 3D, we can interpret the possible outcomes of systems of three equations.

1. Three planes intersect in one point: the system has one unique solution
2. Three planes do not have a common intersection: the system has no solution
3. Three planes have a line in common: the system has many solutions

The three planes can also coincide, then the equations are equivalent.