RIC Point location

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1 Preliminaries

Recall that $E[\sum_i X_i] = \sum_i E[X_i]$, and that when X has possible outcomes $A_1, .., A_z$ then $E[X] = \sum_{i=1}^{z} \Pr[X = A_i]A_i$, where $\Pr[Y]$ denotes the probability of event Y.

2 Bounding Space

Let S be the space used by a (point location structure built on a) vertical decomposition T_n of n segments $s_1, .., s_n$. We are interested in the expected space used by our decomposition, that is $E[S]$.

Theorem 1. *The expected space $E[S]$ used by T_n is $O(n)$.*

Proof. We use backwards analysis. Let T_i be the vertical decomposition after inserting the first i segments. Let K_i be the number of trapezoids in T_i created by s_i.

We have $S = c \sum_i K_i$, for some constant c, and thus $E[S] = c \sum_i E[K_i]$.

$$E[K_i] = \sum_{\Delta \in T_i} \Pr[\Delta \text{ created by } s_i] \cdot 1$$

A given trapezoid Δ is created by one of four segments: the segments $top(\Delta)$, $bottom(\Delta)$, $left(\Delta)$ or $right(\Delta)$.

So, we have

$$\Pr[\Delta \text{ created by } s_i] = Pr[s_i \in \{top(\Delta), bottom(\Delta), left(\Delta), right(\Delta)\}]$$

The probability that the last segment s_i is actually segment $top(\Delta)$ is $1/i$. Similarly, the probability that s_i is actually $bottom(\Delta)$ is also $1/i$. The same for $left(\Delta)$ and $right(\Delta)$. Thus,

$$Pr[s_i \in \{top(\Delta), bottom(\Delta), left(\Delta), right(\Delta)\}] = 4/i$$

Since T_i contains $3i + 1$ trapezoids, we thus have

$$E[K_i] = \sum_{\Delta \in T_i} 4/i = (3i + 1)4/i = O(1).$$

Since we insert n segments, the expected total space is $O(n)$. □

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3 Bounding Query time

We bound the expected query time of a point location query. Let q be the query point q, and let P be the path in the search structure that leads to the leaf trapezoid containing q. We bound the expected length $E[Q]$ of P, and thus the query time.

Theorem 2. *The expected query time is $O(\log n)$.*

Proof. We use backwards analysis. Let Q_i be the number of nodes of the search structure on P created when we inserted segment s_i. We thus have $E[Q] = \sum_{i=1}^{n} E[Q_i]$.

Let D_i be the search structure after inserting the first i segments. We have

$$E[Q_i] = \sum_{v \in (D_i \cap P)} \Pr[v \text{ created by } s_i] \cdot 1$$

Observe that inserting s_i increases the depth by at most three. So it can create at most three nodes on P. Let Δ be the leaf trapezoid of D_i containing q (i.e. the last node of path P). We have:

$$\sum_{v \in (D_i \cap P)} \Pr[v \text{ created by } s_i] \cdot 1 \leq 3 \Pr[\Delta \text{ created by } s_i]$$

As in the analysis of the space, the probability that s_i created the particular trapezoid Δ is $4/i$, and thus $E[Q_i] = 3 \cdot 4/i = 12/i$. The expected length of the path then

$E[Q] = \sum_{i=1}^{n} E[Q_i] = \sum_{i=1}^{n} 12/i = 12 \sum_{i=1}^{n} 1/i = 12H_n$,

where H_n is the n^{th} harmonic number. Since $H_n = O(\log n)$ it follows that the expected query time is $O(\log n)$. □