Final Exam 2022-2023

03 February 2023, 09:00-11:30

This exam has 6 questions for a total of 90 points. You can earn an additional 10 points if you write readable, unambiguous, and technically correct. No statements like "The algorithm runs in $n \log n$." (forgetting the $O(..)$ and forgetting to say that it concerns time), etc. Your final grade will be the number of points divided by 10.

Read every question carefully (!), make sure you understand it, and be sure to answer the question. Answer questions in sufficient but not too much detail. You may **not** use the textbook, or any other notes during the exam. Be sure to put your name on every piece of paper you hand in. Good Luck!

Question 1 (10 points)

For each of the following tasks, state the running time for the best possible algorithm to perform the task. If the algorithm is deterministic, give the worst case running time. If the algorithm is randomized, indicate this and give the expected running time. Use k to denote the output size if applicable.
Stating only the running time is sufficient, no need to explain your answers in detail.

(a) Splitting a simple polygon with n vertices into y-monotone sub polygons.
(b) Reporting all points in an axis parallel cube using a range tree (with fractional cascading) built on a set of n points in $\mathbb{R}^3$.
(c) Computing an Euclidean minimum spanning tree on n points in $\mathbb{R}^2$.
(d) Inserting a line in an arrangement of n lines in $\mathbb{R}^2$.
(e) Given a trapezoidal decomposition built on a polygon P with h holes and a total of n vertices; testing if a query point lies inside P.

Question 2 (10 points)

Let S be a set of n disjoint line segments inside a bounding box R, and let $\mathcal{VD}$ be a vertical decomposition (trapezoidal decomposition) of S inside R. Prove/Give exact bounds[a] on the number of (a) vertices, (b) edges, and (c) trapezoids in $\mathcal{VD}$. Argue why your bounds are correct.

[a] i.e. no big-O notation.

Question 3

Let I be a set of n intervals in $\mathbb{R}^1$.

(a) *(5 points)* Explain how we can use an interval tree to efficiently, i.e. in sublinear time, **count** the number of intervals from I stabbed by a query point $q \in \mathbb{R}^1$. Briefly explain the data structure and the query algorithm.
(b) *(5 points)* How much space does the data structure use, and what query time do you get? Briefly explain your answer.
(c) *(10 points)* Can we store I to answer such counting queries more efficiently? How much space do you need for an optimal query time? Briefly explain your answer.

Question 4 (8 points)

Briefly argue why a simple polygon P with n vertices can be guarded using $\lfloor n/3 \rfloor$ guards.

Question 5

Let P be a set of n points in $\mathbb{R}^2$.

(a) *(3 points)* Give three reasons (e.g. applications or problems) why one would be interested in computing the Voronoi diagram of P.

(b) *(4 points)* A Voronoi diagram can be computed using a sweepline algorithm. Briefly describe the possible types of events in this algorithm.

(c) *(3 points)* How many events do we have of each type? Briefly explain your answer.

(d) *(9 points)* Suppose that each site $p \in P$ has some positive weight w_p (all weights may be different). We can then define the distance function $d^*(p,q) = p_w\|pq\|$ (where $\|pq\|$ denotes the Euclidean distance). Prove that the Voronoi region $V^*(p)$ of p (i.e. the set of all points in $\mathbb{R}^2$ closer to p than to any other site $z \neq p \in P$ according to the d^* distance) may be disconnected. Be precise!

Question 6

Let L be a set of n non-vertical lines in $\mathbb{R}^2$. Let $\mathcal{L}(x) = \min_{\ell \in L} \ell(x)$ denote the so called **lower envelope** of L. The (graph of) the function $\mathcal{L}$ traces the lowest line from the set L as a function of the x-coordinate.

Hint: You may want to include some drawings/sketches to clarify your answers.

(a) *(7 points)* Let $\ell \in L$ be the line realizing $\mathcal{L}(x)$ at some value x, so $p = (x, \ell(x)) = (x, \mathcal{L}(x))$. Describe/characterize the dual ℓ^* of ℓ with respect to the (duals of) p and the other lines in L.

(b) *(4 points)* Let $v = (v_x, \mathcal{L}(v_x))$ be vertex of (the graph of) $\mathcal{L}$ (so v_x is a breakpoint of $\mathcal{L}$). Describe/characterize the dual v^* of v.

(c) *(3 points)* Consider the set X all points $(x', \ell(x'))$ where ℓ realizes $\mathcal{L}$. Describe/characterize the dual X^* of X.

(d) *(9 points)* Briefly describe/sketch an $O(n \log n)$ time algorithm to compute (the graph of) $\mathcal{L}$.