Final Exam 2023-2024

02 February 2024, 13:30-16:00

This exam has 7 questions for a total of 90 points. You can earn an additional 10 points if you write readable, unambiguous, and technically correct. No statements like "The algorithm runs in $n \log n$." (forgetting the $O(..)$ and forgetting to say that it concerns time), etc. Your final grade will be the number of points divided by 10.

Read every question carefully (!), make sure you understand it, and be sure to answer the question. Answer questions in sufficient but not too much detail. You may **not** use the textbook, or any other notes during the exam. Be sure to put your name on every piece of paper you hand in. Good Luck!

Question 1 (10 points)

For each of the following tasks, state the running time for the best possible algorithm to perform the task. If the algorithm is deterministic, give the worst case running time. If the algorithm is randomized, indicate this and give the expected running time. Use k to denote the output size if applicable.
Stating only the running time is sufficient, no need to explain your answers in detail.

(a) Triangulating an x-monotone polygon with n vertices.
(b) Computing a highest point that lies inside a set of n halfplanes (if it exists).
(c) Reporting the number of line segments from S intersected by a vertical query line segment Q, where S is a set of n disjoint horizontal line segments, and $\mathcal{T}$ is a segment tree (in which each node stores its canonical subset in an balanced binary search tree) on S.
(d) Constructing the arrangement of a set of n lines.
(e) Reporting the number of line segments from S intersected by a vertical query line segment Q, where S is a set of n disjoint line segments, and $\mathcal{T}$ is an interval tree (in which each internal node stores two range trees storing the start and endpoints of the segments respectively) on S.

Question 2

(a) *(3 points)* What is the zone of a line?
(b) *(3 points)* Briefly describe/explain the zone theorem, and explain why it is useful.
(c) *(6 points)* Briefly describe two different algorithms to compute the arrangement of a set S of n **line segments**. (Two or three sentences describing each algorithm is sufficient.)
(d) *(2 points)* Why/when would you prefer the first algorithm over the second one?

Question 3

Let P be a set of n points in $\mathbb{R}^2$. Recall the randomized incremental construction algorithm for computing the smallest enclosing disk of P.

(a) *(3 points)* State the subproblem the algorithm has to solve when the newly considered point p_i lies outside the smallest enclosing disk of the $i-1$ points considered so far.
(b) *(2 points)* How efficiently can we solve this subproblem?
(c) *(6 points)* Briefly argue why, in expectation, inserting p_i takes only constant time.

Question 4

Let P be a set of n points in $\mathbb{R}^2$, and let $\mathcal{T}$ be a range tree (without fractional cascading) on the points in P.

(a) *(5 points)* Briefly describe how we can query $\mathcal{T}$ to efficiently report all points from P that lie in an axis-aligned query rectangle R.
(b) *(5 points)* Analyze the query time of the query algorithm from question (a).

Question 5 (10 points)

Let P be a simple polygon with n vertices, and let $\mathcal{T}$ be a triangulation of P. Prove that $\mathcal{T}$ consists of $n-2$ triangles.

Question 6 (15 points)

Let ℓ and m be two lines that intersect in point p, with ℓ steeper than m. Let $\overline{pq}$ be a line segment that lies below (or on) the line ℓ and above (or on) the line m. Finally, let r be a point below m with the same x-coordinate as p.

(a) Draw the above construction in the primal and in the dual plane. Clearly label the following objects and their duals in your drawing: $\overline{pq}, p, q, r, \ell, m$.

(b) Formulate the above paragraph in its dual form using the usual point-line duality. It does not have to be a literal translation, but it should capture all geometric information from the above paragraph.

Question 7

Let P be a set of n points in $\mathbb{R}^2$, and let $(p,q) \in P \times P$ be a closest pair among P; i.e. a pair with the smallest Euclidean distance.

(a) *(10 points)* Prove that $\overline{pq}$ is an edge of the Delaunay triangulation of P.

(b) *(3 points)* How efficiently can we compute the closest pair (p,q)? Briefly explain your answer.

(c) *(7 points)* Suppose that $P = R \cup B$ is actually the disjoint union of a set of "red" points R and a set of "blue" points B. Let $(r,b) \in R \times B$ be a **bichromatic** closest pair, i.e. the closest pair where one point is "red" and one point is "blue". Describe (in roughly one paragraph) how we can efficiently compute such a bichromatic closest pair, and analyze the running time of your solution.