Homework Exam 2 2023-2024

My name and StudentID go here!

Deadline: 13 December 2023, 13:15

This homework exam has 6 questions for a total of 90 points. You can earn 10 additional points by a careful preparation of your hand-in: using a good layout, good spelling, good figures, no sloppy notation, no statements like "The algorithm runs in $n \log n$." (forgetting the $O(..)$ and forgetting to say that it concerns time), etc. Use lemmas, theorems, and figures where appropriate. Your final grade will be the number of points divided by 10. Unless stated otherwise, both randomized and deterministic solutions are allowed. In case you are asked to analyze the running time and your algorithm is randomized, analyze its expected running time.

Question 1 (15 points)

Let $\mathcal{S}$ be a planar subdivision with n vertices, represented as a DCEL. Give pseudo-code for an algorithm that, given a pointer to a vertex v, test if v is incident to a face that is a triangle.

Your algorithm should use the 'Twin', 'NextEdge', 'PrevEdge', etc. fields to navigate (i.e. you cannot assume you can directly access a list of vertices). Describe in a few sentences the main idea of your algorithm and give its running time.

Question 2 (10 points)

Let P be a set of n points in $\mathbb{R}^2$. The *diameter* $diam(P)$ of P is the maximum pairwise distance, $\max_{(p,q)\in P\times P} \|pq\|$. Prove that p and q appear on the convex hull $CH(P)$ of P.

Question 3 (10 points)

Let C be a convex polygon whose vertices $c_1, .., c_n$ are given in clockwise order, i.e. you are given some pointer to an array storing the vertices of C in that order. Give an algorithm that can test in $O(\log n)$ time if a query point $q \in \mathbb{R}^2$ lies inside C. Prove that your algorithm is correct and achieves the desired running time.

Question 4 (8 points)

Prove or disprove: The dual graph of the triangulation of a y-monotone polygon is always a chain, that is, any node in this graph has degree at most two.

Question 5 (20 points)

Let P be a set of n points in $\mathbb{R}^2$, let z be a point that lies strictly in the interior of the convex hull $CH(P)$, and let ρ be a ray (oriented half-line) that starts in z. Develop an expected $O(n)$ time algorithm that, given P, z, and ρ, can find the edge of $CH(P)$ hit by ρ. Prove that your algorithm is correct and achieves the desired running time. You may assume that no three points in P are colinear, and that ρ contains no points of P.

Note that you are *not* given $CH(P)$ itself.

Question 6

Let P be a set of n points in $\mathbb{R}^2$, let $D(c)$ be a unit disk, that is, a disk of radius one and center c, and let $P_c = P \cap D(c)$ be the subset of P that lies in a unit disk centered at c.

1. *(10 points)* Prove that there are at most $O(n^2)$ different sets P_c over all points $c \in \mathbb{R}^2$.
2. *(7 points)* Give a construction that shows that the above bound is tight in the worst case. In other words, show that there can sometimes be $\Omega(n^2)$ different sets P_c.
3. *(10 points)* Let k be a small constant, i.e. $k \in O(1)$. Sketch an $O(n^2 \log n)$ time algorithm that can compute the number of subsets of P of size k that can be covered exactly (i.e. the disk contains no additional points from P) by a unit disk. Two or three paragraphs of description is sufficient; you do not have to prove correctness or give the full analysis.