Range searching and kd-trees

Computational Geometry

Utrecht University

Introduction

Introduction

Database queries

Databases

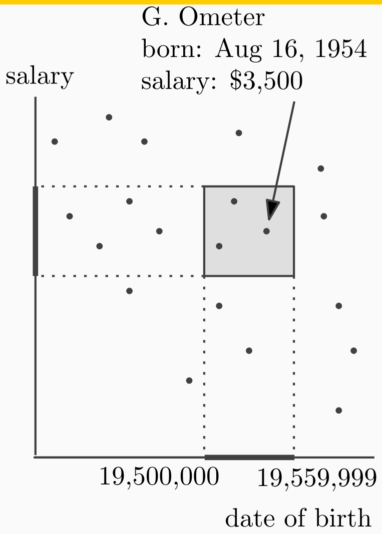
Databases store records or objects

Personnel database: Each employee has a name, id code, date of birth, function, salary, start date of employment, . . .

Fields are textual or numerical

Database queries

A database query may ask for all employees with age between a_1 and a_2, and salary between s_1 and s_2



Database queries

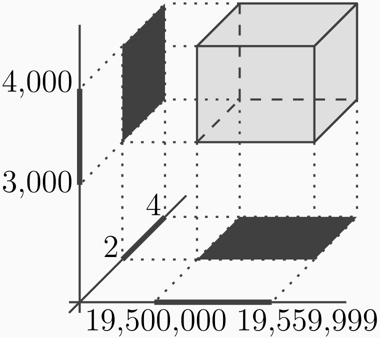
When we see numerical fields of objects as coordinates, a database stores a point set in higher dimensions

Exact match query: Asks for the objects whose coordinates match query coordinates exactly

Partial match query: Some but not all coordinates are specified

Range query: Asks for the objects whose coordinates lie in a specified query range (interval)

Example of a 3-dimensional (orthogonal) range query: children in $[2\,,\,4]$, salary in $[3000\,,\,4000]$, date of birth in $[19,500,000\ ,\ 19,559,999]$



Data structures

Idea of data structures

- Representation of structure, for convenience (like DCEL)
- Preprocessing of data, to be able to solve future questions really fast (sub-linear time)

A (search) data structure has a storage requirement, a query time, and a construction time (and an update time)

Introduction

1D range trees

1D range query problem

1D range query problem: Preprocess a set of n points on the real line such that the ones inside a 1D query range (interval) can be reported fast

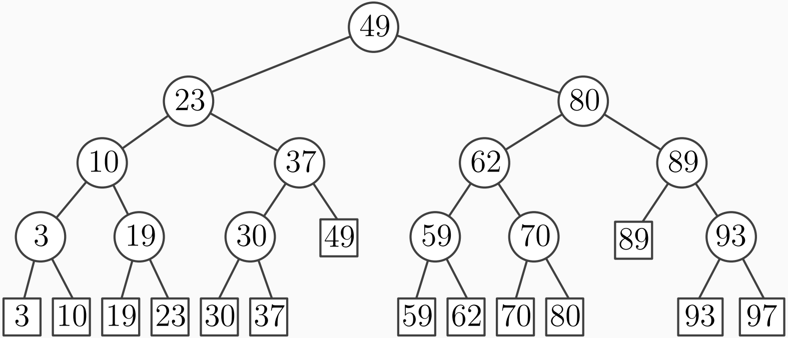
The points $p_1, \ldots, p_n$ are known beforehand, the query $[x, x']$ only later

A solution to a query problem is a data structure description, a query algorithm, and a construction algorithm

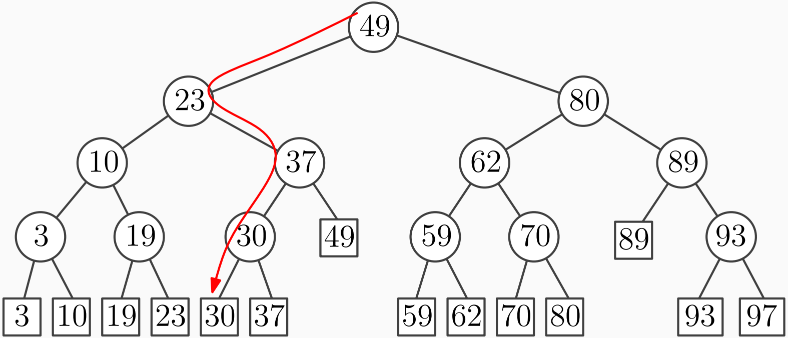
Question: What are the most important factors for the *efficiency* of a solution?

Balanced binary search trees

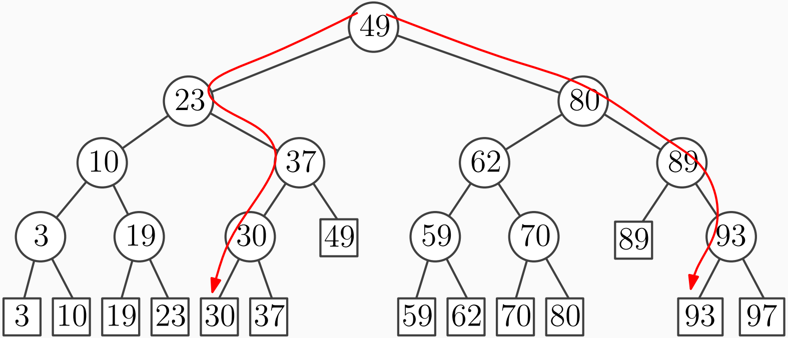
A balanced binary search tree with the points in the leaves



The search path for 25

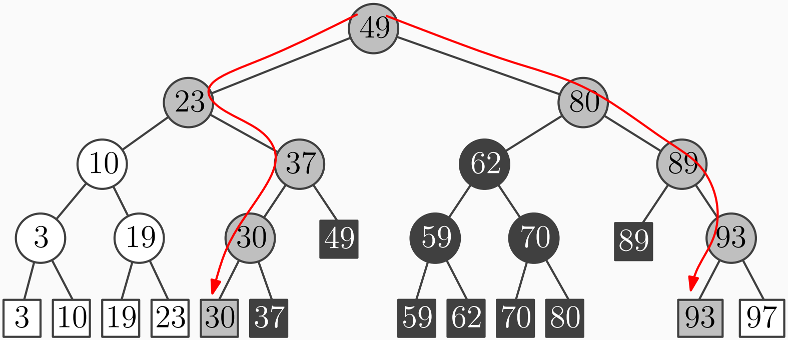


The search paths for 25 and for 90



Example 1D range query

A 1-dimensional range query with $[25, 90]$



Node types for a query

Three types of nodes *for a given query*:

- **White nodes:** never visited by the query
- **Grey nodes:** visited by the query, unclear if they lead to output
- **Black nodes:** visited by the query, whole subtree is output

Question: What query time do we hope for?

Node types for a query

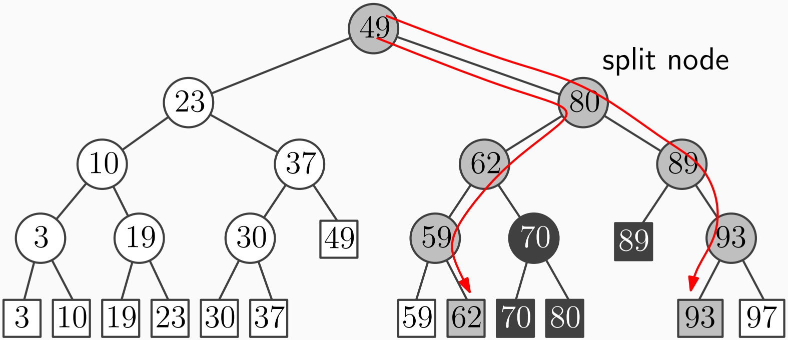
The query algorithm comes down to what we do at each type of node

Grey nodes: use query range to decide how to proceed: to not visit a subtree (pruning), to report a complete subtree, or just continue

Black nodes: traverse and enumerate all points in the leaves

Example 1D range query

A 1-dimensional range query with $[61, 90]$



1D range query algorithm

Algorithm 1DRangeQuery($\mathcal{T}, [x : x']$)

$\nu_{\text{split}} \leftarrow$FindSplitNode($\mathcal{T}, x, x'$)
if ν_{split} is a leaf
    then Check if the point in ν_{split} must be reported.
    else $\nu \leftarrow lc(\nu_{\text{split}})$
        while ν is not a leaf
            do if $x \leq x_\nu$
                then ReportSubtree($rc(\nu)$)
                        $\nu \leftarrow lc(\nu)$
                else $\nu \leftarrow rc(\nu)$
        Check if the point stored in ν must be reported.
        $\nu \leftarrow rc(\nu_{\text{split}})$
        Similarly, follow the path to x', and ...

Query time analysis

The efficiency analysis is based on counting the numbers of nodes visited for each type

- **White nodes:** never visited by the query; no time spent
- **Grey nodes:** visited by the query, unclear if they lead to output; time determines dependency on n
- **Black nodes:** visited by the query, whole subtree is output; time determines dependency on k, the output size

Query time analysis

Grey nodes: they occur on only two paths in the tree, and since the tree is balanced, its depth is $O(\log n)$

Black nodes: a (sub)tree with m leaves has $m-1$ internal nodes; traversal visits $O(m)$ nodes and finds m points for the output

The time spent at each node is $O(1) \;\Rightarrow\; O(\log n + k)$ query time

Storage requirement and preprocessing

A (balanced) binary search tree storing n points uses $O(n)$ storage

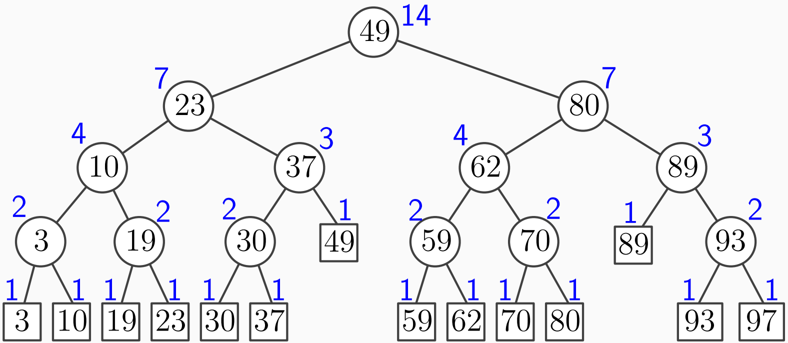
A balanced binary search tree storing n points can be built in $O(n)$ time after sorting, so in $O(n \log n)$ time overall
(or by repeated insertion in $O(n \log n)$ time)

Result

Theorem: A set of n points on the real line can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any 1D range query can be answered in $O(\log n + k)$ time, where k is the number of answers reported

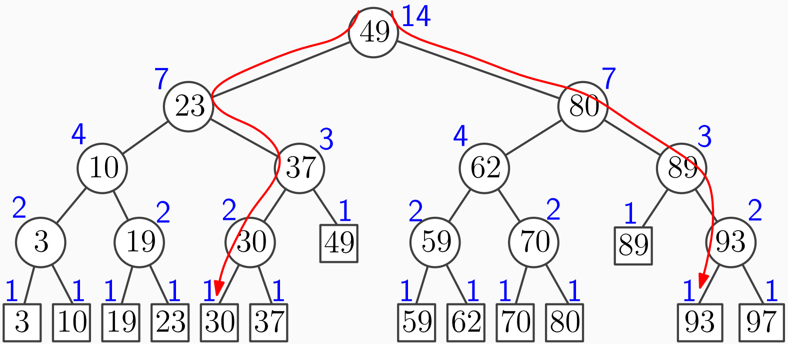
Example 1D range counting query

A 1-dimensional range tree for **range counting queries**



Example 1D range counting query

A 1-dimensional range counting query with $[25, 90]$



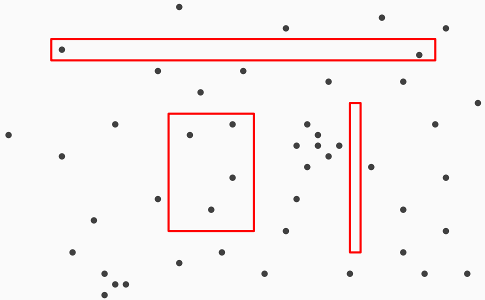
Result

Theorem: A set of n points on the real line can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any 1D range counting query can be answered in $O(\log n)$ time

Note: The number of points does not influence the output size so it should not show up in the query time

Kd-trees

Range queries in 2D



Question: Why can't we simply use a balanced binary tree in x-coordinate?

Or, use one tree on x-coordinate and one on y-coordinate, and query the one where we think querying is more efficient?

Kd-trees

Kd-trees

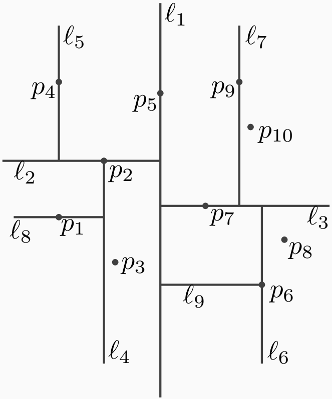
Kd-trees

Kd-trees, the idea: Split the point set alternatingly by x-coordinate and by y-coordinate

split by x-coordinate: split by a vertical line that has half the points left and half right

split by y-coordinate: split by a horizontal line that has half the points below and half above

Kd-trees


ℓ_1
ℓ_5
ℓ_7
p_4
p_5
p_9
p_{10}
ℓ_2
p_2
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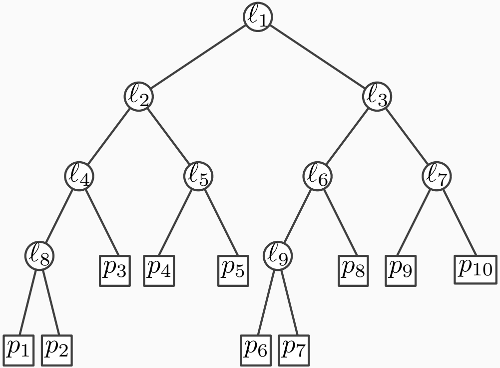
ℓ_3
ℓ_8
p_1
p_8
p_3
ℓ_9
p_6
ℓ_4
ℓ_6


ℓ_1
ℓ_2
ℓ_3
ℓ_4
ℓ_5
ℓ_6
ℓ_7
ℓ_8
p_3
p_4
p_5
ℓ_9
p_8
p_9
p_{10}
p_1
p_2
p_6
p_7

Kd-tree construction

Algorithm BuildKdTree($P, depth$)

1. **if** P contains only one point
2. **then return** a leaf storing this point
3. **else** **if** $depth$ is even
4. **then** Split P with a vertical line ℓ through the median x-coordinate into P_1 (left of or on ℓ) and P_2 (right of ℓ)
5. **else** Split P with a horizontal line ℓ through the median y-coordinate into P_1 (below or on ℓ) and P_2 (above ℓ)
6. $v_{\text{left}} \leftarrow$ BuildKdTree($P_1, depth + 1$)
7. $v_{\text{right}} \leftarrow$ BuildKdTree($P_2, depth + 1$)
8. Create a node v storing ℓ, make v_{left} the left child of v, and make v_{right} the right child of v.
9. **return** v

Kd-tree construction

The median of a set of n values can be computed in $O(n)$ time (randomized: easy; worst case: much harder)

Let $T(n)$ be the time needed to build a kd-tree on n points

$$T(1) = O(1)$$

$$T(n) = 2 \cdot T(n/2) + O(n)$$

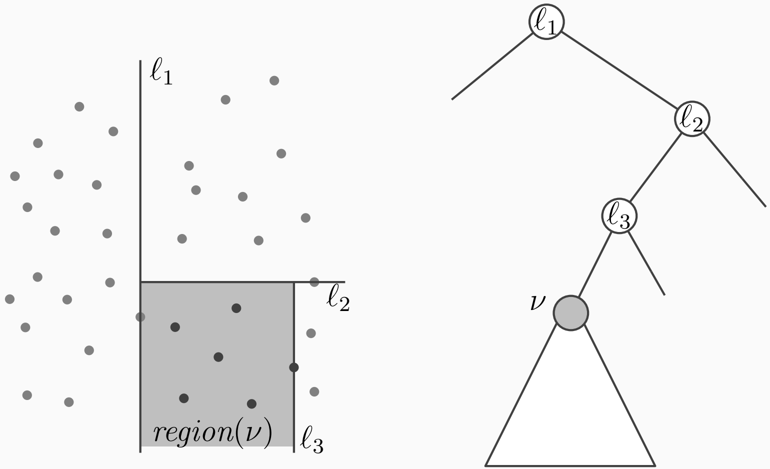
A kd-tree can be built in $O(n \log n)$ time

Question: What is the storage requirement?

Kd-trees

Querying in kd-trees

Kd-tree regions of nodes



Kd-tree regions of nodes

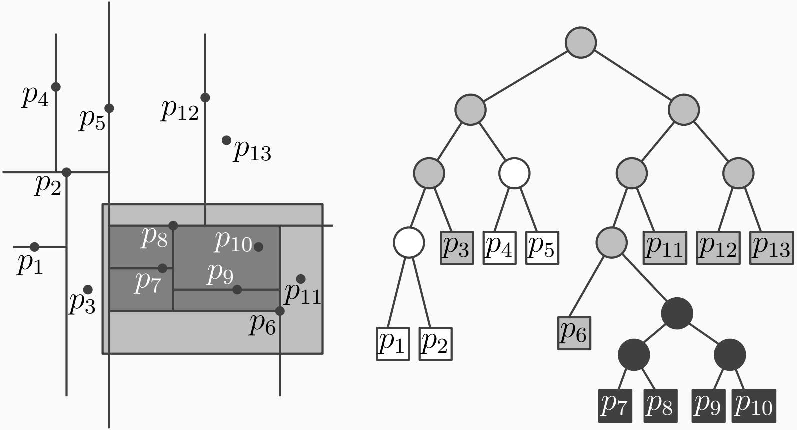
How do we know $region(v)$ when we are at a node v?

Option 1: store it explicitly with every node

Option 2: compute it on-the-fly, when going from the root to v

Question: What are reasons to choose one or the other option?

Kd-tree querying



Kd-tree querying

Algorithm SearchKdTree(v, R)

Input. The root of (a subtree of) a kd-tree, and a range R

Output. All points at leaves below v that lie in the range.

if v is a leaf
  then Report the point stored at v if it lies in R
  else **if** $region(lc(v))$ is fully contained in R
    then ReportSubtree($lc(v)$)
    else **if** $region(lc(v))$ intersects R
      then SearchKdTree($lc(v), R$)
  if $region(rc(v))$ is fully contained in R
    then ReportSubtree($rc(v)$)
    else **if** $region(rc(v))$ intersects R
      then SearchKdTree($rc(v), R$)

Kd-tree querying

Question: How about a range *counting* query?

How should the code be adapted?

Kd-trees

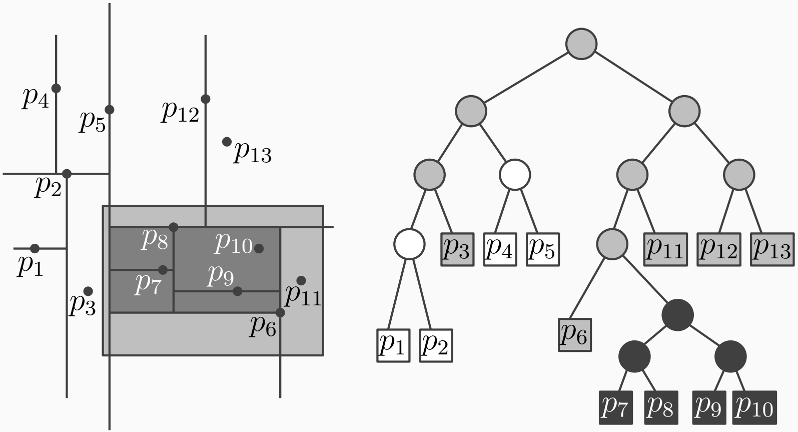
Kd-tree query time analysis

Kd-tree query time analysis

To analyze the query time of kd-trees, we use the concept of white, grey, and black nodes

- **White nodes:** never visited by the query; no time spent
- **Grey nodes:** visited by the query, unclear if they lead to output; time determines dependency on n
- **Black nodes:** visited by the query, whole subtree is output; time determines dependency on k, the output size

Kd-tree query time analysis

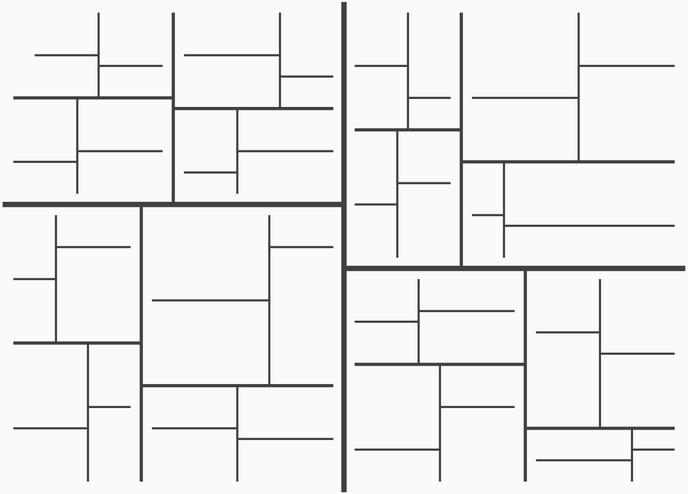


Kd-tree query time analysis

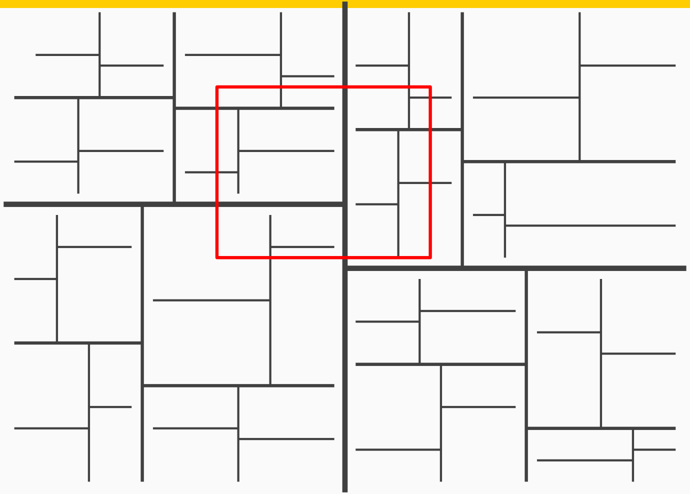
White, grey, and black nodes with respect to $region(v)$:

- **White node v:** R does not intersect $region(v)$
- **Grey node v:** R intersects $region(v)$, but $region(v) \not\subseteq R$
- **Black node v:** $region(v) \subseteq R$

Kd-tree query time analysis

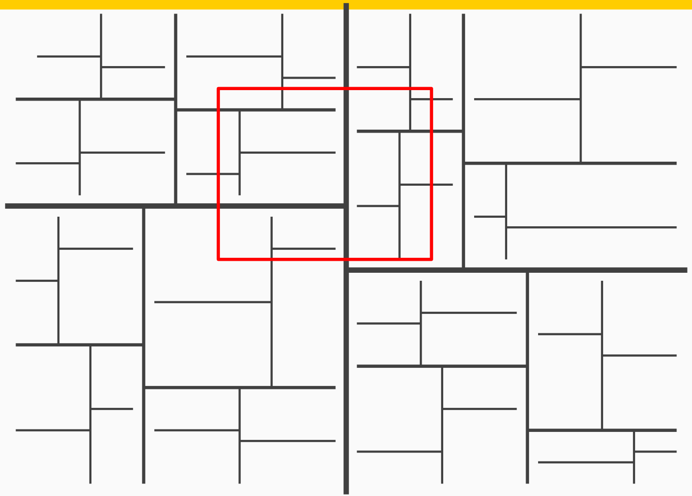


Kd-tree query time analysis



Question: How many grey and how many black *leaves*?

Kd-tree query time analysis



Question: How many grey and how many black *nodes*?

Kd-tree query time analysis

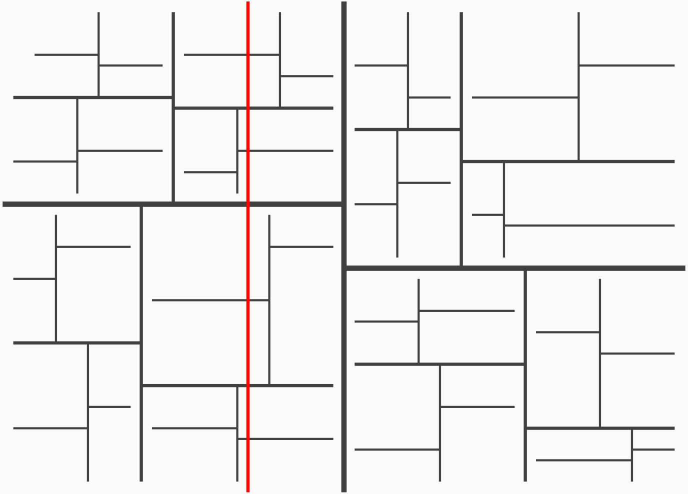
Grey node v: R intersects $region(v)$, but $region(v) \not\subseteq R$

It implies that the boundaries of R and $region(v)$ intersect

Advice: If you don't know what to do, simplify until you do

Instead of taking the boundary of R, let's analyze the number of grey nodes if the query is with a vertical line ℓ

Kd-tree query time analysis



Question: How many grey and how many black *nodes*?

Kd-tree query time analysis

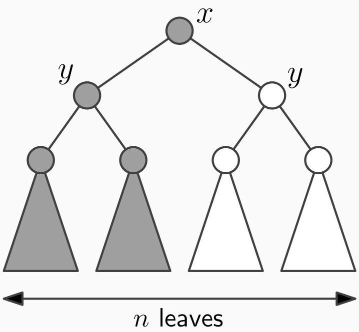
We observe: At every vertical split, ℓ is only to one side, while at every horizontal split ℓ is to both sides

Let $G_x(n)$ be the number of grey nodes in a kd-tree on n points whose root node splits on x (vertically).

Let $G_y(n)$ be the number of grey nodes in a kd-tree on n points whose root node splits on y (horizontally).

$$G_x(n) = \begin{cases} G_y(n/2) + 1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases} \qquad G_y(n) = \begin{cases} 2G_x(n/2) + 1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Kd-tree query time analysis



Kd-tree query time analysis

Let $G_x(n)$ be the number of grey nodes in a kd-tree on n points whose root node splits on x (vertically).

So, we get:

$$G_x(n) = \begin{cases} 2G_x(n/4) + 2 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Kd-tree query time analysis

Let $G_x(n)$ be the number of grey nodes in a kd-tree on n points whose root node splits on x (vertically).

So, we get:

$$G_x(n) = \begin{cases} 2G_x(n/4) + O(1) & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Question: What does this recurrence solve to?

Kd-tree query time analysis

Use the Master-Theorem:

$$T(n) = aT(n/b) + f(n)$$

let $c = \log_b a$, let $\varepsilon > 0$

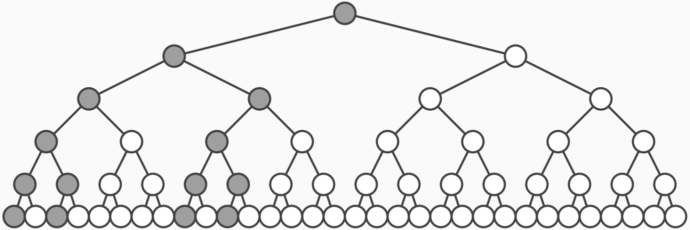
case 1: $f(n) \in O(n^{c-\varepsilon})$ then $T(n) = O(n^c)$.

case 2: ...

case 3 ...

Here $f(n) = O(1)$ and $c = \log_4 2 = 1/2$. Therefor $G_x(n) = O(n^{1/2}) = O(\sqrt{n})$.

Kd-tree query time analysis



The grey subtree has unary and binary nodes

Kd-tree query time analysis

The depth is $\log n$, so the binary depth is $\frac{1}{2} \cdot \log n$

Important: The logarithm is base-2

Counting only binary nodes, there are

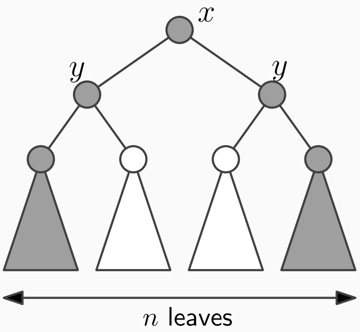
$$2^{\frac{1}{2} \cdot \log n} = 2^{\log n^{1/2}} = n^{1/2} = \sqrt{n}$$

Every unary grey node has a unique binary parent (except the root), so there are at most twice as many unary nodes as binary nodes, plus 1

Kd-tree query time analysis

The number of grey nodes if the query were a vertical line is $O(\sqrt{n})$

For a horizontal line we get



Kd-tree query time analysis

The number of grey nodes if the query were a vertical line is $O(\sqrt{n})$

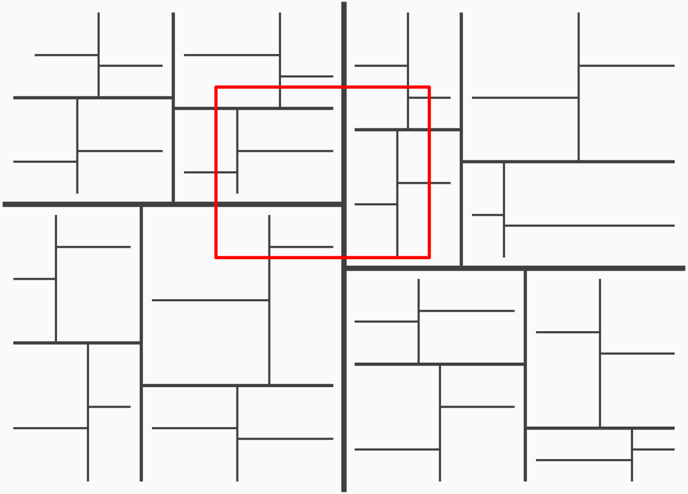
For a horizontal line we get

$$G(n) = \begin{cases} 2G(n/4) + 3 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

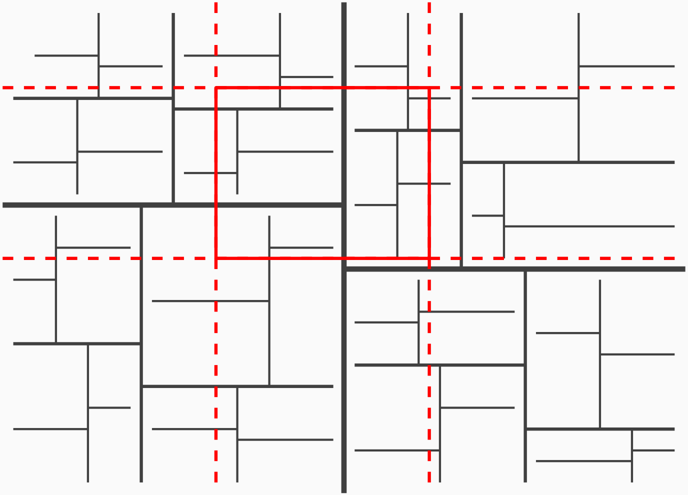
Which also solves to $O(\sqrt{n})$.

How about a query rectangle?

Kd-tree query time analysis



Kd-tree query time analysis



Kd-tree query time analysis

The number of grey nodes for a query rectangle is at most the number of grey nodes for two vertical and two horizontal lines, so it is at most $4 \cdot O(\sqrt{n}) = O(\sqrt{n})$!

For black nodes, reporting a whole subtree with k leaves, takes $O(k)$ time (there are $k-1$ internal black nodes)

Result

Theorem: A set of n points in the plane can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any 2D range query can be answered in $O(\sqrt{n} + k)$ time, where k is the number of answers reported

For range counting queries, we need $O(\sqrt{n})$ time

Efficiency

n	$\log n$	$\sqrt{n}$
4	2	2
16	4	4
64	6	8
256	8	16
1024	10	32
4096	12	64
1.000.000	20	1000

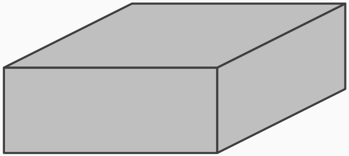
Kd-trees

Higher-dimensional kd-trees

Higher dimensions

A 3-dimensional kd-tree alternates splits on x-, y-, and z-coordinate

A 3D range query is performed with a box



Higher dimensions

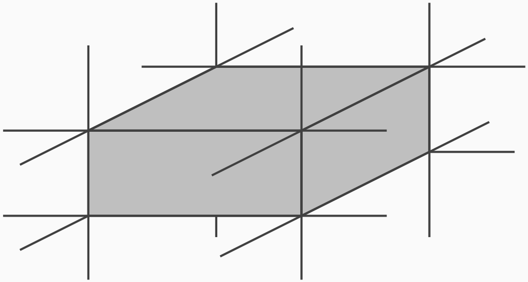
The construction of a 3D kd-tree is a trivial adaptation of the 2D version

The 3D range query algorithm is exactly the same as the 2D version

The 3D kd-tree still requires $O(n)$ storage if it stores n points

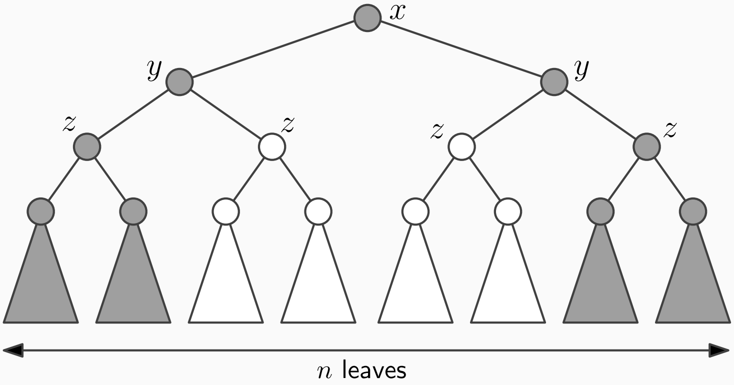
Higher dimensions

How does the query time analysis change?



Intersection of B and $region(v)$ depends on intersection of facets of B $\Rightarrow$ analyze by axes-parallel planes (B has no more grey nodes than six planes)

Higher dimensions



Kd-tree query time analysis

Let $G_3(n)$ be the number of grey nodes for a query with an axes-parallel plane in a 3D kd-tree

$$G_3(1) = 1$$

$$G_3(n) = 4 \cdot G_3(n/8) + O(1)$$

Question: What does this recurrence solve to?

Question: How many leaves does a perfectly balanced binary search tree with depth $\frac{2}{3}\log n$ have?

Result

Theorem: A set of n points in d-space can be preprocessed in $O(n \log n)$ time into a data structure of $O(n)$ size so that any d-dimensional range query can be answered in $O(n^{1-1/d} + k)$ time, where k is the number of answers reported