Lecture 11: Arrangements and Duality

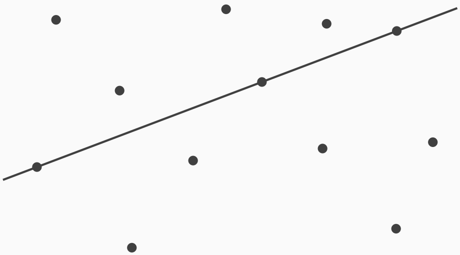
Computational Geometry

Utrecht University

Introduction

Three Points on a Line

Question: In a set of n points, are there 3 points on a line?



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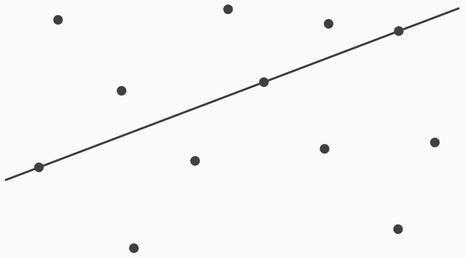

Naive algorithm: tests all triples in $O(n^3)$ time

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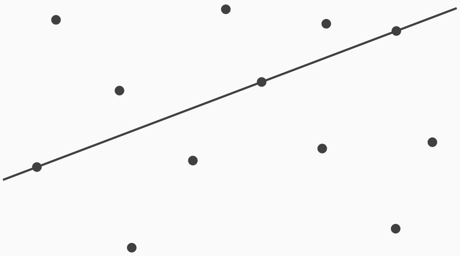

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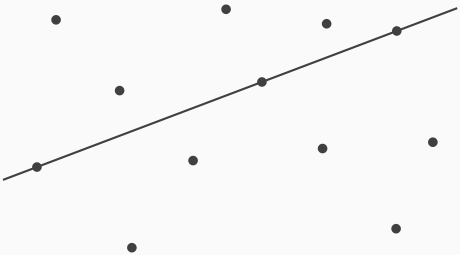

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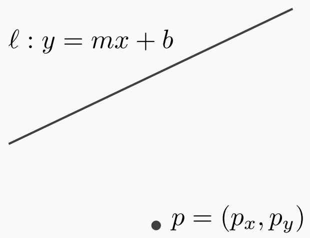
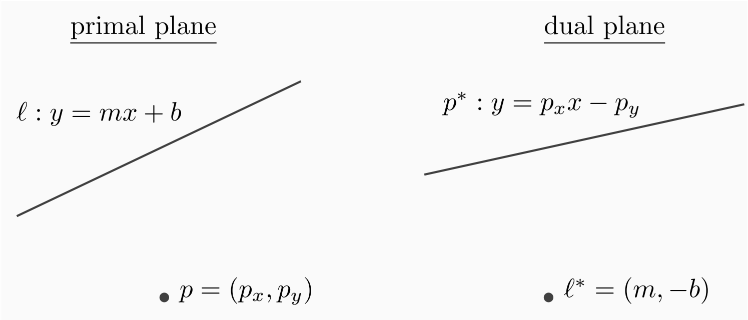
Note: other motivation in chapter 8 of the book

Duality

Duality



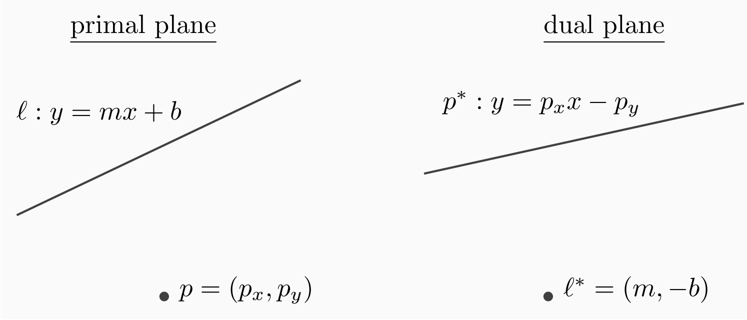
Duality



point $p = (p_x, p_y) \mapsto$ line $p^* : y = p_x x - p_y$

line $\ell : y = mx + b \mapsto$ point $\ell^* = (m, -b)$

Duality

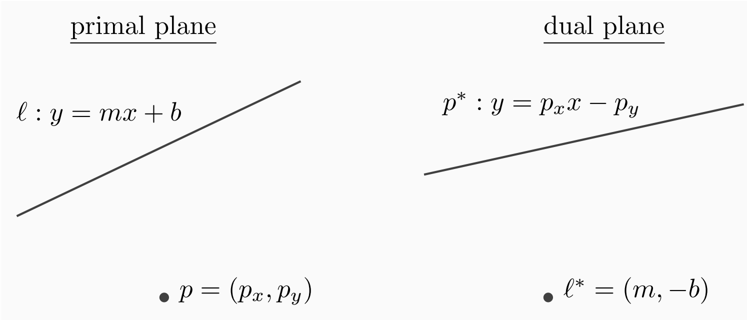


$$\text{point } p = (p_x, p_y) \mapsto \text{line } p^* : y = p_x x - p_y$$

$$\text{line } \ell : y = mx + b \mapsto \text{point } \ell^* = (m, -b)$$

Note: self inverse $(p^*)^* = p, \quad (\ell^*)^* = \ell$

Duality



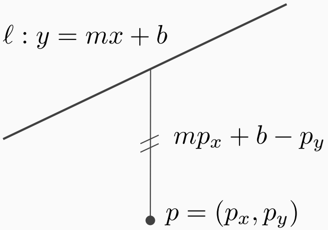
point $p = (p_x, p_y) \mapsto$ line $p^* : y = p_x x - p_y$

line $\ell : y = mx + b \mapsto$ point $\ell^* = (m, -b)$

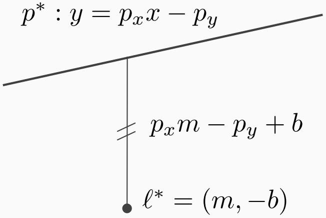
Note: does not handle vertical lines

Duality

primal plane

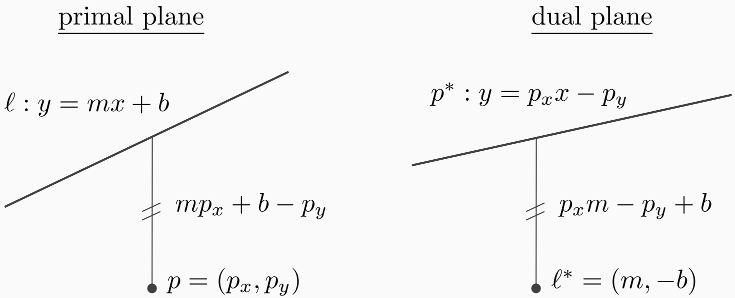


dual plane



Duality preserves vertical distances

Duality



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$\Rightarrow$ incidence preserving: $p \in \ell$ if and only if $\ell^* \in p^*$

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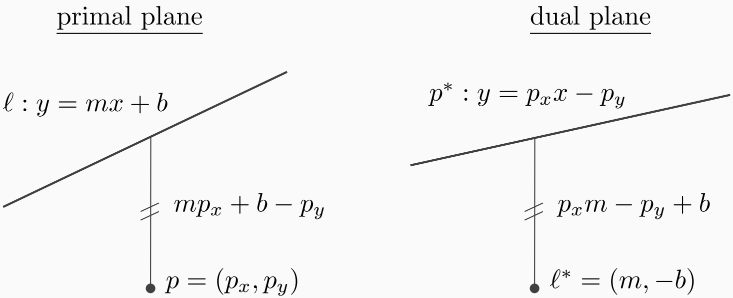

Duality preserves vertical distances

$\Rightarrow$ incidence preserving: $p \in \ell$ if and only if $\ell^* \in p^*$

$\Rightarrow$ order preserving: p lies below ℓ if and only if ℓ^* lies below p^*

It can be applied to other objects, like segments

Duality

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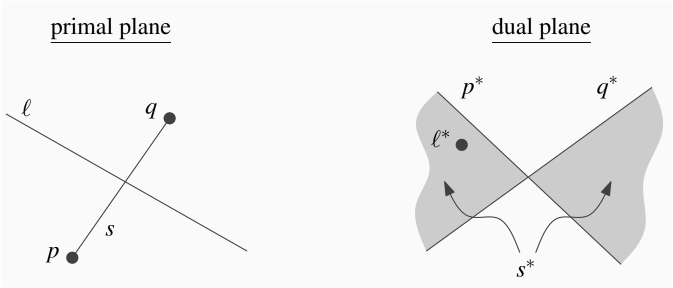


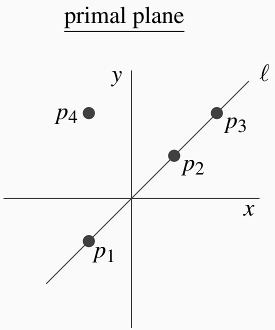
The dual of a segment is a double wedge

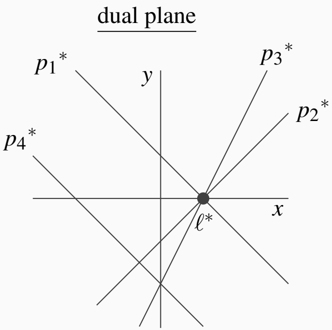
Question: What line would dualize to a point in the right part of the double wedge?

Usefulness of Duality

Why use duality? It gives a new perspective!

Detecting three points on a line dualizes to detecting three lines intersecting in a point





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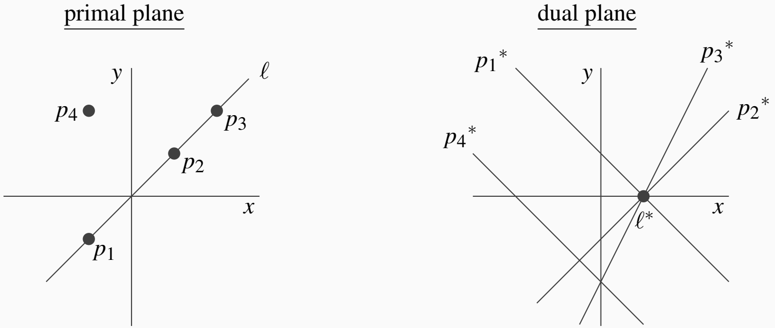
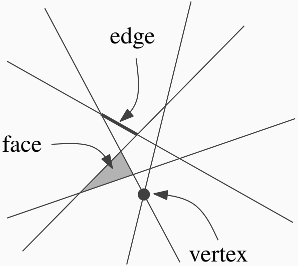


Next we use arrangements

Arrangements

Arrangement $\mathcal{A}(L)$: subdivision induced by a set of lines L

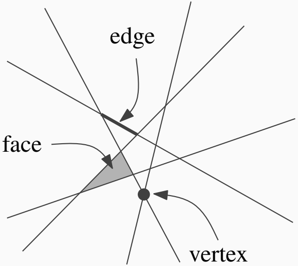
- consists of *faces*, *edges* and *vertices* (some unbounded)



Arrangements of Lines

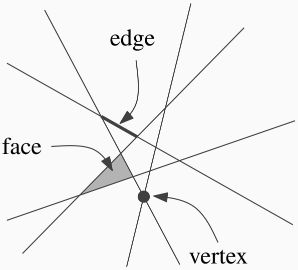
Arrangement $\mathcal{A}(L)$: subdivision induced by a set of lines L

- consists of *faces, edges* and *vertices* (some unbounded)
- arrangements consist of other geometric objects too, like line segments, circles, higher-dimensional objects



Combinatorial Complexity:

- $\leq n(n-1)/2$ vertices



Arrangements of Lines

Combinatorial Complexity:

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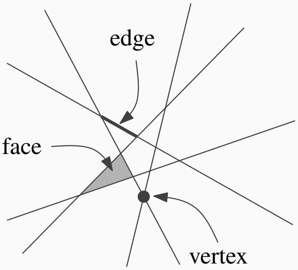
- $\leq n^2$ edges



Combinatorial Complexity:

- $\leq n(n-1)/2$ vertices
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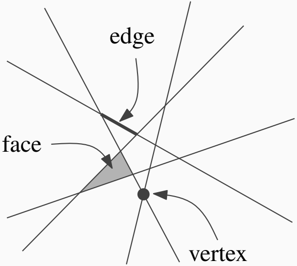
  add lines incrementally

$$1 + \sum_{i=1}^{n} i = n(n+1)/2 + 1$$



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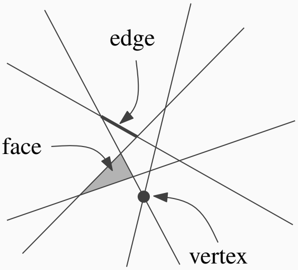
- equality holds in *simple* arrangements



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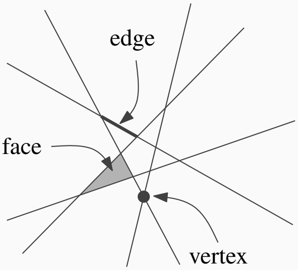
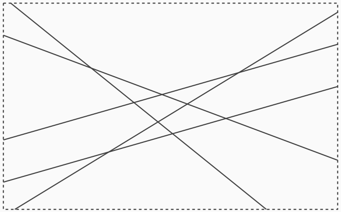
Overall $O(n^2)$ complexity



Constructing Arrangements

Goal: Compute $\mathcal{A}(L)$ in bounding box in DCEL representation



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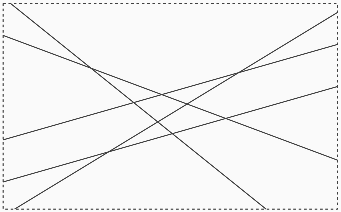

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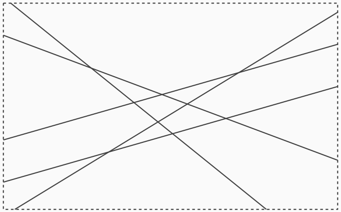
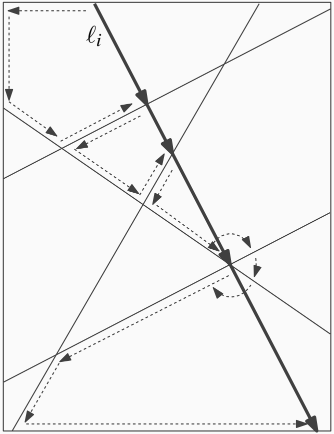


- plane sweep for line segment intersection: $O((n+k)\log n) = O(n^2 \log n)$
- faster: incremental construction

Arrangements

Incremental Construction



Algorithm ConstructArrangement(L)

Input. Set L of n lines

Output. DCEL for $\mathcal{A}(L)$ in $\mathcal{B}(L)$

1. Compute bounding box $\mathcal{B}(L)$
2. Construct DCEL for subdivision induced by $\mathcal{B}(L)$
3. **for** $i \leftarrow 1$ **to** n
4.   **do** insert ℓ_i

Incremental Construction

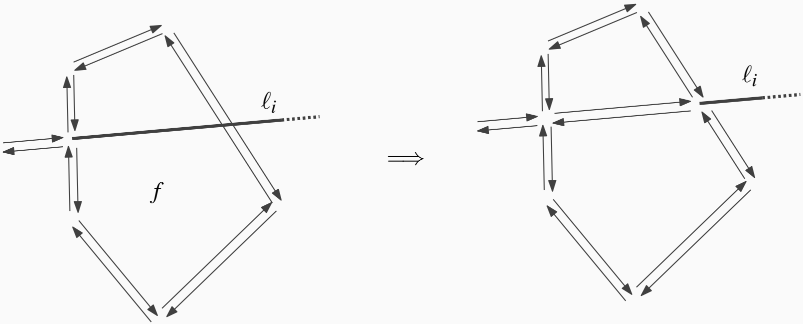
Algorithm ConstructArrangement(L)

Input. A set L of n lines in the plane

Output. DCEL for subdivision induced by $\mathcal{B}(L)$ and the part of $\mathcal{A}(L)$ inside $\mathcal{B}(L)$, where $\mathcal{B}(L)$ is a suitable bounding box

1. Compute a bounding box $\mathcal{B}(L)$ that contains all vertices of $\mathcal{A}(L)$ in its interior
2. Construct DCEL for the subdivision induced by $\mathcal{B}(L)$
3. **for** $i \leftarrow 1$ **to** n
4. **do** Find the edge e on $\mathcal{B}(L)$ that contains the leftmost intersection point of ℓ_i and $\mathcal{A}_i$
5. $f \leftarrow$ the bounded face incident to e
6. **while** f is not the unbounded face, that is, the face outside $\mathcal{B}(L)$
7. **do** Split f, and set f to be the next face intersected by ℓ_i

Incremental Construction



Face split:

Runtime analysis:

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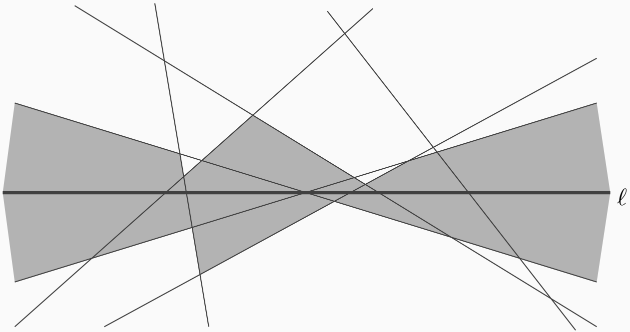
Zone Theorem

The zone of a line ℓ in an arrangement $\mathcal{A}(L)$ is the set of faces of $\mathcal{A}(L)$ whose closure intersects ℓ



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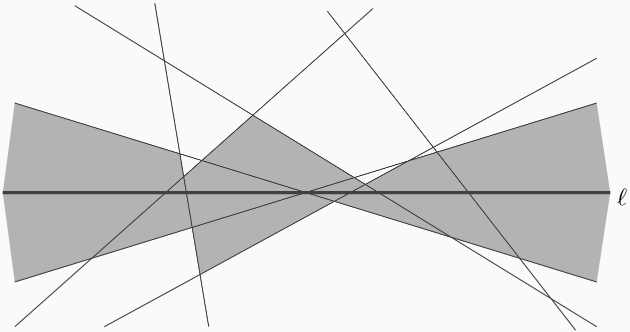

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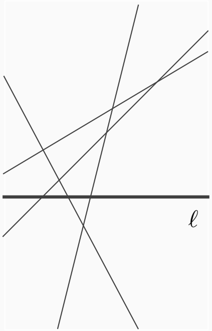

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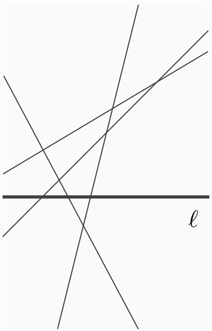

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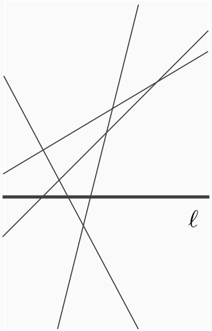

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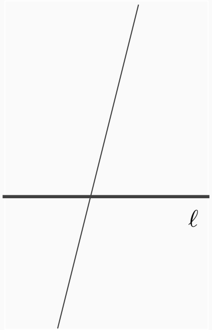

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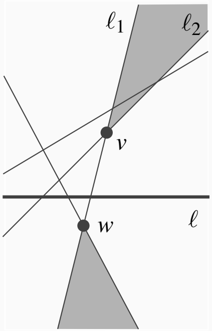

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- We show by induction on m that this at most $5m$:
  - $m = 1$: trivially true
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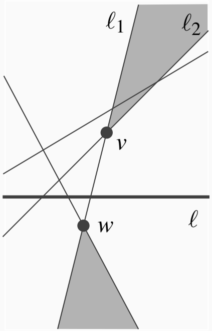
    $5(m-1) + 5 = 5m$



Run time analysis:

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In total $O(n^2)$

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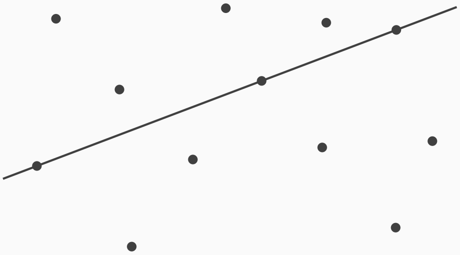
3 Points on a Line



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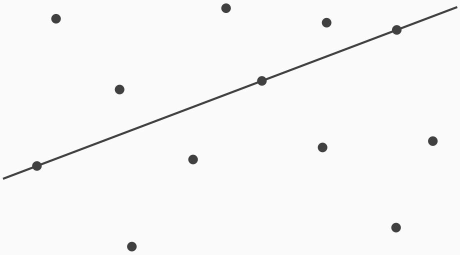


Algorithm:

- run incremental construction algorithm for dual problem
- stop when 3 lines pass through a point

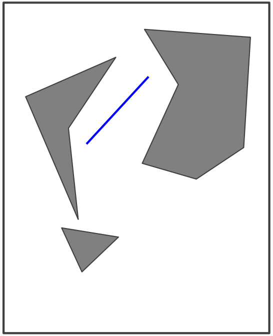
Run time: $O(n^2)$

Arrangements

Motion Planning

Example: Motion Planning

Where can the rod move by translation (no rotations) while avoiding obstacles?



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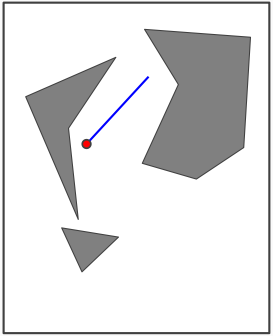

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Where can the rod move by translation (no rotations) while avoiding obstacles?

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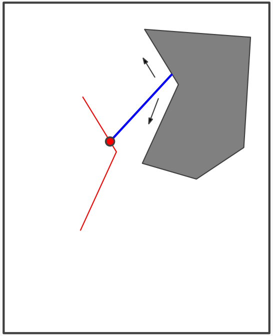
- shrink rod to a point,

  expand obstacles accordingly:

  locus of semi-free placements



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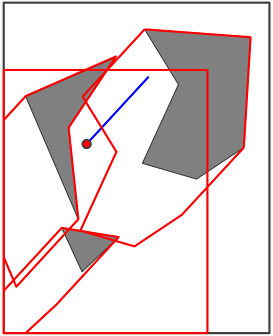

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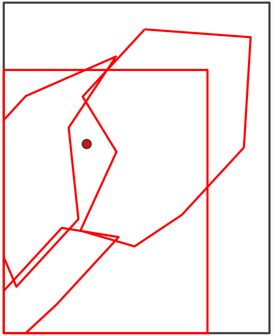
  expand obstacles accordingly:
  locus of semi-free placements
- reachable configurations:
  cell of initial configuration in
  arrangement of line segments

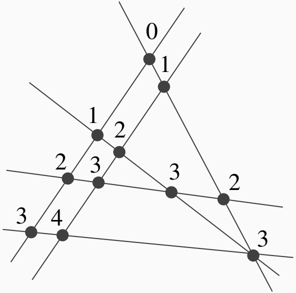


Arrangements

k-Levels

k-levels in Arrangements

The level of a point in an arrangement of lines is the number of lines strictly above it



k-levels in Arrangements

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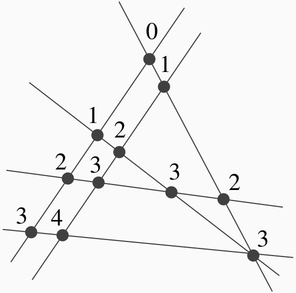
Open problem: What is the complexity of k-levels?



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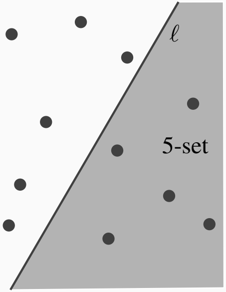
Dual problem: What is the number of k-sets in a point set?



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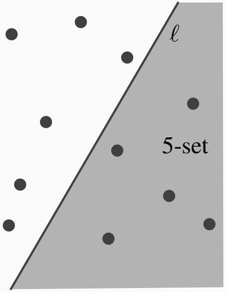
Known bounds:

- Erdös et al. ’73: $\Omega(n \log k)$ and $O(nk^{1/2})$



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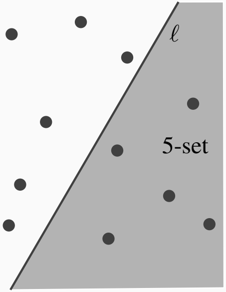
- Dey '97: $O(nk^{1/3})$



Three dimensions

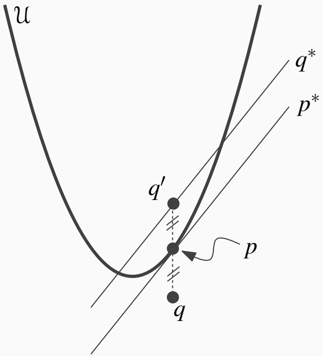
In 3D, we have point-plane duality; lines dualize to other lines

An arrangement induced by n planes in 3D has complexity $O(n^3)$

Deciding whether a set of points in 3D has four or more co-planar points can be done in $O(n^3)$ time (dualize and construct the arrangement)

A geometric interpretation:

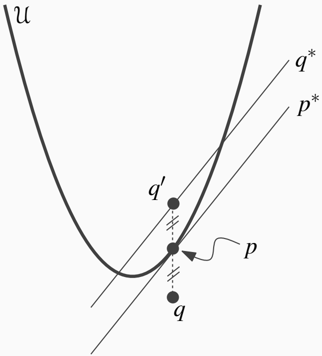
- parabola $\mathcal{U} : y = x^2/2$



More Duality

A geometric interpretation:

- parabola $\mathcal{U} : y = x^2/2$
- point $p = (p_x, p_y)$ on $\mathcal{U}$



A geometric interpretation:

- parabola $\mathcal{U} : y = x^2/2$
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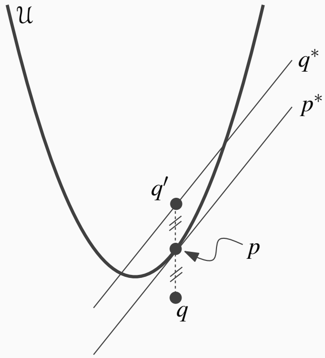
- derivative of $\mathcal{U}$ at p is p_x, i.e., p^* has same slope as the tangent line



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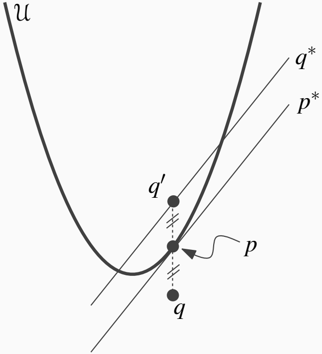
- the tangent line intersects y-axis at $(0, -p_x^2/2)$



More Duality

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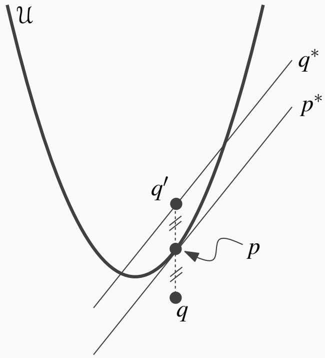
- derivative of $\mathcal{U}$ at p is p_x, i.e., p^* has same slope as the tangent line
- the tangent line intersects y-axis at $(0, -p_x^2/2)$
- $\Rightarrow$ p^* is the tangent line at p



Summary

Duality is a useful tool to reformulate certain problems on points in the plane to lines in the plane, and vice versa

Dualization of line segments is especially useful

Arrangements, zones of lines in arrangements, and levels in arrangements are useful concepts in computational geometry

All of this exists in three and higher dimensional spaces too