UU Graphics

academic year 2013/14 – 4th period

Theoretical Assignment #1:
Basic Math and Linear Algebra

Apr 29 2014

General Remark

The theoretical assignments are very important to gain *procedural skills*; you will learn how to work with the concepts discussed in the lecture: How to perform calculations, how to prove properties, and how to apply theoretical models to concrete problems. The skills developed in the theory tutorials are complementary to the practicals – you need to do both in order to master the topic.

Working on the practical assignments and attending the tutorials will be crucial for passing the two exams. The assignments discussed here will be similar to the ones in the exams. If you have not worked on such problems & assignments enough, chances of passing will in most cases be very low.

Of course, there might be exceptions. For example, if you are minoring or even majoring in mathematics, parts of the assignments might be concerned with material you are already very familiar with. In such cases, it might be wise to allocate more time to develop your practical skills. However, if you are not yet familiar with the topics of these assignments, we very strongly recommend working through all of them carefully and allocating a substantial fraction of your study time to this task.

That said, we hope that you enjoy the solving the problems below and those of the future tutorials!

Solutions - general remark:

As in previous years, we do not provide full solutions for all details but focus on the challenging parts and the main ideas. This is also to make sure that you try to solve all assignments on your own – you can only see the full solution if you work it out yourself or in groups. If you have difficulties, visit the tutorials and make use of the supervision and advice provided. We are making considerable efforts to provide help and individual assistance with your questions and problems, and you can greatly benefit from this!

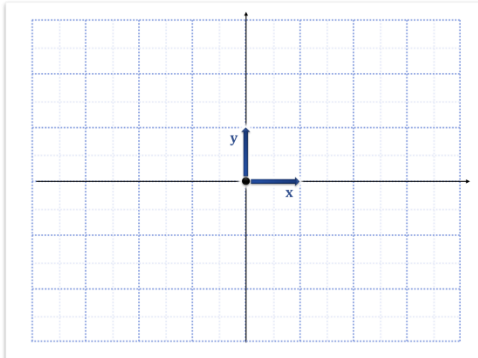


Fig.1: Coordinate grid – draw the situation in a coordinate frame

Assignment #1: *Very basic vector calculations*

BASIC
basic topics
study completely

Assume we are given three vectors from $\mathbb{R}^2$:

$$\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix}2\\2\end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix}1\\0\end{pmatrix}$$

a) Draw the vectors on a sheet of paper (it is good to draw a grid first, as shown in Fig. 1 above).

b) Compute the sum of $\mathbf{a}$ and $\mathbf{b}$. Draw the result.

c) Scale the vector a by the factors $1{,}2, -1$ and $\sqrt{2}$. Draw the results.

d) Determine $\mathbf{b} - \mathbf{c}$ graphically, using the rule from the lecture: first, join the starting points of $\mathbf{b}$ and $\mathbf{c}$, then draw an arrow from the tip of $\mathbf{c}$ to the tip of $\mathbf{b}$, which gives you the result.
Similarly, visualize the result of adding $\mathbf{b}$ and $-\mathbf{c} = (-\mathbf{1}) \cdot \mathbf{c}$, which gives the exact same result. (Keep in mind that arrows are compared as arrows; the starting point does not matter, only the arrow itself.) Make sure to fully understand this, and ask the tutors if anything remains unclear!

e) Compute and draw $2(\mathbf{a} + \mathbf{c})$ and $2\mathbf{a} + 2\mathbf{c}$. Visualize how this gives the same result (distributive rule).

f) Build a linear combination $\mathbf{v}$ of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, i.e., $\mathbf{v} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}$.
Choose suitable factors $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and draw the result.

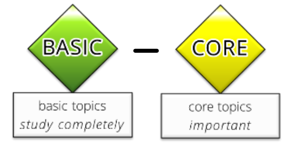
Remarks: This first assignment is only meant to familiarize yourself with geometric vectors. There is no big insight here. If you already took vector algebra in high-school, this should be very easy.

Solution:

Parts a,b,c,d,e are really elementary. Just draw it.

f): for example, draw the vector $1 \cdot \mathbf{a} + 0.5 \cdot \mathbf{b} + 2 \cdot \mathbf{c} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}, + \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix}$.

Assignment #2: *Linear Combinations and Spans*



Assume we are given four vectors from $\mathbb{R}^2$:

$$\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix}2\\2\end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix}0\\1\end{pmatrix}$$

a) Which vectors are in $\operatorname{span}\{\mathbf{a}\}$? Sketch the set of points in your drawing (recommendation: start a new drawing for assignment 2)

b) Find a linear combination of $\mathbf{c}$ and $\mathbf{d}$ that lies in $\operatorname{span}\{\mathbf{a}\}$.
Rephrased more formally: Determine some $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1\mathbf{c} + \lambda_2\mathbf{d} \in \operatorname{span}\{\mathbf{a}\}$.

c) In general, which coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ can you choose such that $\lambda_1\mathbf{c} + \lambda_2\mathbf{d} \in \operatorname{span}\{\mathbf{a}\}$. Can you find a general rule?

d) The span of all four vectors, $\operatorname{span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is the entire 2D plane $\mathbb{R}^2$. Enumerate all subsets of $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ that still span all of $\mathbb{R}^2$. Not all combinations will work. Can you explain why some subsets cannot span all of $\mathbb{R}^2$?

e) A minimal set of vectors that span all of $\mathbb{R}^2$ can serve as a coordinate system for describing vectors numerically. “Minimal” means that we leave out any vector that can be already linearly combined by the other ones (i.e., we want to have a linearly independent set that spans $\mathbb{R}^2$). We do this repeatedly until no more vector can be omitted. Such a set is also called a *basis* of $\mathbb{R}^2$.
Your assignment: Provide at least two *different* coordinate systems (bases) of $\mathbb{R}^2$. Determine the coordinates of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in those bases, i.e., compute coefficients $\lambda_1, \lambda_2 \in \mathbb{R}$ such that a linear combination of your basis vectors with those coefficients creates our example vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

f) Prove for one such example basis that it is linearly independent. You can use any of the criteria listed in the lecture to show this.

Remark: Linear combinations are the fundamental operation that can be performed with vectors. This assignment should help you in gaining a better intuition in how vectors can be linearly combined from other vectors. Coordinate systems or “bases” are special sets of vectors that both span the space and are also linearly independent.

Solution:

a) The vector b is in span{a} (it lies on the same line, or b=2a). This is not possible for c,d:

$$\lambda\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \text{ implies that } \lambda = 1 \text{ and } \lambda = 0\text{, which is a contradiction.}$$

b) $\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \mathbf{a}$

c) $\lambda_1\begin{pmatrix}1\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\end{pmatrix} = \mu\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = \mu$ (solve system of equations).

d) The combination a,b is not sufficient. any other combination with at least two vectors works.

Explanation: **a** and **b** lie on the same line; the span is therefore not capturing **c**,**d**. See also part (a).

e) We go with the following two bases:

$$B_1 = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}, \qquad B_2 = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

For B_1, the representation is already given. For B_2, we get:
$\mathbf{a} =_{B_2} \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{b} =_{B_2} \begin{pmatrix}0\\2\end{pmatrix}, \mathbf{c} =_{B_2} \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{d} =_{B_2} \begin{pmatrix}-1\\1\end{pmatrix}$. This is easy to see – just form the linear combinations.

f) We show that the null-vector can only be trivially constructed:

$$\lambda_1 \begin{pmatrix}1\\0\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{matrix} 1\cdot\lambda_1 + 0\cdot\lambda_2 = 0 \\ \text{and } 0\cdot\lambda_1 + 1\cdot\lambda_2 = 0 \end{matrix} \Rightarrow \begin{matrix} \lambda_1 = 0 \\ \text{and } \lambda_2 = 0 \end{matrix}$$

Assignment #3: *Properties of Vector Operations*

This assignment comes with a longer explanation to recap some important definitions from the lecture. The actual assignment is rather short.

CORE

core topics *important*

Introduction
We now want to prove the algebraic properties of vector operations that were mentioned in the lecture. In the following, we are considering a d-dimensional real vector space $\mathbb{R}^d$. As explained in the lecture, we define such a vector space as the set of all columns of d real numbers:

$$\mathbb{R}^d := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \middle| \, x_1, x_2, \dots, x_d \in \mathbb{R} \right\}$$

Notation: A vector is denoted by a bold letter (small caps) such as "$\mathbf{x}$" and the individual components are denoted by small caps of the same letter in italics, such as $x_1, x_2, \dots, x_d$. The subscript is the row-index within the column. Sometimes, we also use x, y or x, y, z to denote the components for the special case of 2D or 3D vectors, but this scheme is not useful if we have many different vectors that all need different characters to be distinguished.

Hint: On paper, it is often useful to denote vectors with arrows above the letter (as it is difficult to perform bold-print without serious harm to the paper). So we would sometimes write $\vec{x}$ for $\mathbf{x}$. The individual components $x_1, x_2, \dots, x_d$ do not get arrows here.

Definition of vector space operations: We define the addition of vectors as component-wise addition of the real-numbers that constitute them. Assume we are given two vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix}$$

The sum of these two is given by:

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_d + y_d \end{pmatrix}$$

The scalar-vector-product of vector $\mathbf{x}$ with a real number λ is defined as:

$$\lambda \cdot \mathbf{x} = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} := \begin{pmatrix} \lambda \cdot x_1 \\ \lambda \cdot x_2 \\ \vdots \\ \lambda \cdot x_d \end{pmatrix}$$

Remark: We use an implementation-oriented, slightly less comprehensive definition of a vector space here. In pure mathematics, a vector space is defined by axioms. For "finite-dimensional" vector spaces, i.e., those that have a finite basis (which are the only ones we are interested in in this lecture), one can prove that they can always be represented using a scheme as shown above. A representation as arrays of numbers can be derived by just establishing such a basis (consisting of *d* basis vectors) and proving that all bases have the same number of vectors. The coordinates of the basis vectors (the factors needed for linearly combining a given vectors) are then written in a column, and are added and multiplied by the rules above. Therefore, there is no difference in practice whether we follow the direct definition above or the axiomatic one (unless we want to describe infinite-dimensional vector spaces, such as spaces of continuous functions, as very briefly shown in the lecture; as I said above, we won't do such things in this course).

Your Assignment

Now we should show the following properties using the definition above. This should be quite easy; we can just reduce it to properties of the real numbers, which we consider as given (all algebraic rules with-in $\mathbb{R}$ are known to be true).

Show the following properties:

(a) Vector addition:

- (A1) For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- (A2) For all $\mathbf{u}, \mathbf{v} \in \mathrm{V}$: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (A3) There exists a null-vector „$\mathbf{0}$"$\in \mathrm{V}$ such that for all $\mathbf{v} \in \mathrm{V}$: $\mathbf{v} + \mathbf{0} = \mathbf{v}$
- (A4) For all $\mathbf{v} \in \mathrm{V}$ there exists an additive inverse of $\mathbf{v}$, denoted by **-$\mathbf{v}$** $\in \mathrm{V}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$

(b) Compatibility with scalar multiplication:

- (S1) For all $\mathbf{v} \in \mathrm{V}, \lambda, \mu \in \mathbb{R}$: $\lambda(\mu\mathbf{u}) = \lambda\mu(\mathbf{u})$
- (S2) For all $\mathbf{v} \in \mathrm{V}$: $1 \cdot \mathbf{v} = \mathbf{v}$
- (S3) For all $\mathbf{v}, \mathbf{w} \in \mathrm{V}, \lambda \in \mathbb{R}$: $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
- (S4) For all $\mathbf{v} \in \mathrm{V}, \lambda, \mu \in \mathbb{R}$: $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

One more remark: In the formal mathematical definition, these properties are *the axioms* that actually define a vector space. In this assignment, we show that our more specific definition has all of these properties required for a general abstract vector space. In other words, our definition is a special case of the general concept.

Solutions:

A1, A2: The first two cases have been shown in the lecture.

A3: The vector $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ is the null vector: $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 + 0 \\ \vdots \\ v_d + 0 \end{pmatrix} = \mathbf{v}$

A4 – S4: similar idea – this is elementary.

Assignment #4: *Geometric Transformations*

Imagine we are given a set of points in $\mathbb{R}^d$ (such as the bunny silhouette on the lecture slides). Develop a formula by which the bunny can be scaled by a factor of λ and the center of the scaling (the point that does not move for any scale factor) is the point $\mathbf{p} \in \mathbb{R}^d$.

Solutions:

As one student already suggested in the lecture in response to this question, we have to move the center of scaling to the origin first, then scale, and then move it back. The following transformation will do the trick:

$$f(\mathbf{x}) = \lambda(\mathbf{x} - \mathbf{p}) + \mathbf{p}$$