Theoretical Assignment #2: Linear Algebra II

May 06 2014

Assignment #1: *Raytracing! (at least, we will need this later for raytracing…)*

BASIC
basic topics
study completely

We are given the following triangle in $\mathbb{R}^3$:

$$T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| 0 \leq \mu_1, \mu_2 \leq 1, \mu_1 + \mu_2 \leq 1 \right\}$$

In addition, we are given three rays, specified below in their parametric ray equation:

$$R_1 = \left\{ \begin{pmatrix} 0.25 \\ 0.25 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

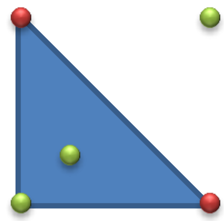
$$R_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

$$R_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

First, sketch the triangle T and the rays R_1, R_2, R_3 in a sketch/drawing. It is recommended to visualize the situation by projecting on the x-y plane (dropping z or using parallel projection for the z-direction). Which of the three rays R_1, R_2, R_3 intersects triangle T, and why? Can you formally compute the parameters μ_1, μ_2, λ to check your result?

When computing the intersection formally, what kind of system of equations do you need to solve?

Solution:



Only R_1 hits; R_2 misses because it starts at too large depth; R_3 misses the triangle completely.

$$\mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.25 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mu_1 &= 0.25 \\ \mu_2 &= 0.25 \\ -\lambda &= -2 \end{aligned} \Rightarrow \begin{aligned} \mu_1 &= 0.25 \\ \mu_2 &= 0.25 \\ \lambda &= 2 \end{aligned} \Rightarrow \text{intersects (all within bounds)}$$

$$\mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= 0 \\ -\lambda &= 2 \end{aligned} \Rightarrow \begin{aligned} \mu_1 &= 0 \\ \mu_2 &= 0 \\ \lambda &= -2 \end{aligned} \Rightarrow \text{does not intersect } (\lambda<0)$$

$$\mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mu_1 &= 1 \\ \mu_2 &= 1 \\ -\lambda &= -2 \end{aligned} \Rightarrow \begin{aligned} \mu_1 &= 1 \\ \mu_2 &= 1 \\ \lambda &= 2 \end{aligned} \Rightarrow \text{does not intersect } (\mu_1 + \mu_2 > 1)$$

Assignment #2: *What are these objects?*

Please classify the following objects as (i) a linear space, (ii) an affine space, (iii) something different from the two. Explain why.

a) The set $\{\lambda_1\mathbf{u} + \lambda_2\mathbf{v} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors

b) The set $\{(\lambda_1 + \lambda_3)\mathbf{u} + (\lambda_2 + \lambda_3)\mathbf{v} | \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors

c) The set $\{\lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_1\lambda_2(\mathbf{u} + \mathbf{v}) | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors

d) The set $\{\lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \mathbf{p} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v}, \mathbf{p} \in \mathbb{R}^3$ are fixed vectors

e) The set $\{\lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \mathbf{u} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors

Solution:

Linear spaces are always also affine spaces; I am not listing this explicitly.

a) is a linear space (span of vectors)

b) is a linear space (span of vectors): $(\lambda_1 + \lambda_3)\mathbf{u} + (\lambda_2 + \lambda_3)\mathbf{v} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \lambda_3(\mathbf{u} + \mathbf{v})$

c) is non-linear and non-affine (product of λ_1, λ_2)

d) is an affine (not a linear) space if $\mathbf{p} \neq \mathbf{0}$.

e) $\{\lambda_1\mathbf{u} + \lambda_2\mathbf{v} + \mathbf{u} | \lambda_1, \lambda_2 \in \mathbb{R}\} = \{(\lambda_1 + 1) \cdot \mathbf{u} + \lambda_2\mathbf{v} + \mathbf{u} | \lambda_1, \lambda_2 \in \mathbb{R}\} = \{\lambda_1 \cdot \mathbf{u} + \lambda_2\mathbf{v} + \mathbf{u} | \lambda_1, \lambda_2 \in \mathbb{R}\}$ because $\{x + 1 | x \in \mathbb{R}\} = \mathbb{R}$. Hence, this is always a linear space (no matter how $\mathbf{u}, \mathbf{v}$ are chosen).

Assignment #3: *Practicing Vector Algebra*

Proof the following equations by applying the algebraic rules for vector spaces. Instructions: apply algebraic rules to the left-hand side until you get the result on the right-hand side.

a) First, just compute the expression

$$3\left(\begin{pmatrix}1\\1\end{pmatrix} + 4\left(\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}2\\1\end{pmatrix}\right)\right) + 2\begin{pmatrix}2\\1\end{pmatrix}$$

Solution

$$\begin{pmatrix}3\\3\end{pmatrix} + 12\begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} + \begin{pmatrix}36\\36\end{pmatrix} + \begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}43\\41\end{pmatrix}$$

b) Now show the following: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be vectors, and $\mu, \lambda \in \mathbb{R}$ be real numbers. Then:

$$\lambda\big(\mathbf{u} + \mu(\mathbf{v} + \mathbf{w})\big) + 2\mathbf{w} = \lambda\mathbf{u} + \lambda\mu\mathbf{v} + (\lambda\mu + 2)\mathbf{w}$$

In order to prove this result, you can use either the definition of vector spaces from the lecture (as d-tupels) or the axioms listed in Assignment 3 of Tutorial 1 (if you want to go with “pure mathematics”;

in this case, cite all of the rules that you are using to convert the left-hand-side into the right-hand one).

Vector space axioms (from Tutorial 1)
(a) Vector addition:

- R1 For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathrm{V}$: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
- R2 For all $\mathbf{u}, \mathbf{v} \in \mathrm{V}$: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
- R3 There exists a null-vector „$\mathbf{0}$"$\in \mathrm{V}$ such that for all $\mathbf{v} \in \mathrm{V}$: $\mathbf{v}+\mathbf{0}=\mathbf{v}$
- R4 For all $\mathbf{v} \in \mathrm{V}$ there exists an additive inverse of $\mathbf{v}$, denoted by **-$\mathbf{v}$** $\in \mathrm{V}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$

(b) Compatibility with scalar multiplication:

- R5 For all $\mathbf{v} \in \mathrm{V}, \lambda, \mu \in \mathbb{R}$: $\lambda(\mu\mathbf{u})=\lambda\mu(\mathbf{u})$
- R6 For all $\mathbf{v} \in \mathrm{V}$: $1 \cdot \mathbf{v}=\mathbf{v}$
- R7 For all $\mathbf{v}, \mathbf{w} \in \mathrm{V}, \lambda \in \mathbb{R}$: $\lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}$
- R8 For all $\mathbf{v} \in \mathrm{V}, \lambda, \mu \in \mathbb{R}$: $(\lambda+\mu)\mathbf{v}=\lambda\mathbf{v}+\mu\mathbf{v}$

Table 1: Vector space axioms recap, with numbering.

Solution: (numbering the rules from Assignment 1 by R_1-R_8, as show in Table 1)

$$
\begin{aligned}
\lambda\big(\mathbf{u}+\mu(\mathbf{v}+\mathbf{w})\big)+2\mathbf{w} &=_{\mathrm{R7}} \lambda\mathbf{u}+\lambda\mu(\mathbf{v}+\mathbf{w})+2\mathbf{w} \\
&=_{R7} \lambda\mathbf{u}+\lambda\mu\mathbf{v}+\lambda\mu\mathbf{w}+2\mathbf{w} \\
&=_{R8} \lambda\mathbf{u}+\lambda\mu\mathbf{v}+(\lambda\mu+2)\mathbf{w}
\end{aligned}
$$

c) Check that (a) yields the same result.

It will.

d) We now practice the sum notation. First, just explain why the following equation holds (by applying the respective rules from Tutorial 1, Assignment 3):

$$(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2)\cdot(\mu_1+\mu_2)=\lambda_1\mu_1\mathbf{v}_1+\lambda_2\mu_1\mathbf{v}_2+\lambda_1\mu_2\mathbf{v}_1+\lambda_2\mu_2\mathbf{v}_2$$

Solution:

$$
\begin{aligned}
(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2)\cdot(\mu_1+\mu_2) &=_{R8} \mu_1(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2)+\mu_2(\lambda_1\mathbf{v}_1+\lambda_2\mathbf{v}_2) \\
&=_{R7} (\lambda_1\mu_1\mathbf{v}_1+\lambda_2\mu_1\mathbf{v}_2)+(\lambda_1\mu_2\mathbf{v}_1+\lambda_2\mu_2\mathbf{v}_2) \\
&=_{R1} \lambda_1\mu_1\mathbf{v}_1+\lambda_2\mu_1\mathbf{v}_2+\lambda_1\mu_2\mathbf{v}_1+\lambda_2\mu_2\mathbf{v}_2
\end{aligned}
$$

e) Now, explain why this formula is true:

$$\left(\sum_{i=1}^{n}\lambda_i\mathbf{v}_i\right)\cdot\left(\sum_{i=1}^{n}\mu_i\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\mu_j\mathbf{v}_i$$

where $\lambda_1,\dots,\lambda_n\in\mathbb{R},\ \mu_1,\dots,\mu_k\in\mathbb{R}, \mathbf{v}_1,\dots,\mathbf{v}_n\in\mathbb{R}^d$.

Solution:

$$\begin{aligned}\left(\sum_{i=1}^{n}\lambda_i\mathbf{v}_i\right)\cdot\left(\sum_{i=1}^{n}\mu_i\right) &=_{R2}\left(\sum_{i=1}^{n}\mu_i\right)\left(\sum_{i=1}^{n}\lambda_i\mathbf{v}_i\right)\\ &=_{R7'}\sum_{i=1}^{n}\left(\mu_i\sum_{j=1}^{n}\lambda_j\mathbf{v}_j\right)\\ &=_{R7'}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\lambda_i\mu_j\mathbf{v}_i\right)\\ &=\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\mu_j\mathbf{v}_i\end{aligned}$$

We have to show a variant of the distributive law R7 – that it holds for larger sums and not just for pairs. This can be done by induction:

$$\text{to be shown} - \ R7'\colon \lambda(\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_\mathrm{n}) =_! \lambda\mathbf{v}_1+\lambda(\mathbf{v}_2+\cdots+\mathbf{v}_\mathrm{n})$$

We know that $\lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}$. Now consider $\lambda(\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_\mathrm{n})$. This can be written as $\lambda(\mathbf{v}_1+(\mathbf{v}_2+\cdots+\mathbf{v}_\mathrm{n}))$ by applying rule R1 *n*-1 times (actually, this should also be shown by induction, but this is so obvious, I'll skip it). By rule R7, we obtain $\lambda\mathbf{v}_1+\lambda(\mathbf{v}_2+\cdots+\mathbf{v}_\mathrm{n})$. We can now apply the argument recursively (induction) to obtain the result $R7'$.

Assignment #4: *Bonus assignment (in reaction to the question in the lecture :-))*

We want to build a convex polygon in 2D (we will use a quadrilateral, not a 7-sided Heptagon).

a) Consider a rectangle in $\mathbb{R}^2$, spanned by the following four points as corners:

$$\mathbf{p}_1=\begin{pmatrix}0\\0\end{pmatrix},\qquad \mathbf{p}_2=\begin{pmatrix}2\\0\end{pmatrix},\qquad \mathbf{p}_3=\begin{pmatrix}1\\1\end{pmatrix},\qquad \mathbf{p}_4=\begin{pmatrix}0\\1\end{pmatrix}$$

Describe this quadrilateral as intersection of four half spaces: The whole 2D plane can be parametrized by its canonical coordinate system as follows:

$$\mathbb{R}^2=\left\{x\cdot\begin{pmatrix}1\\0\end{pmatrix}+y\cdot\begin{pmatrix}0\\1\end{pmatrix}\,\middle|\,x,y\in\mathbb{R}\right\}$$

Formulate four linear inequalities $H_i(x, y) \equiv \left[a_1^{(i)}x + a_2^{(i)}y + a_3^{(i)} \leq 0\right]$ such that their conjunction (applying logical "and") describes exactly the points inside the quadrilateral.

Solution:

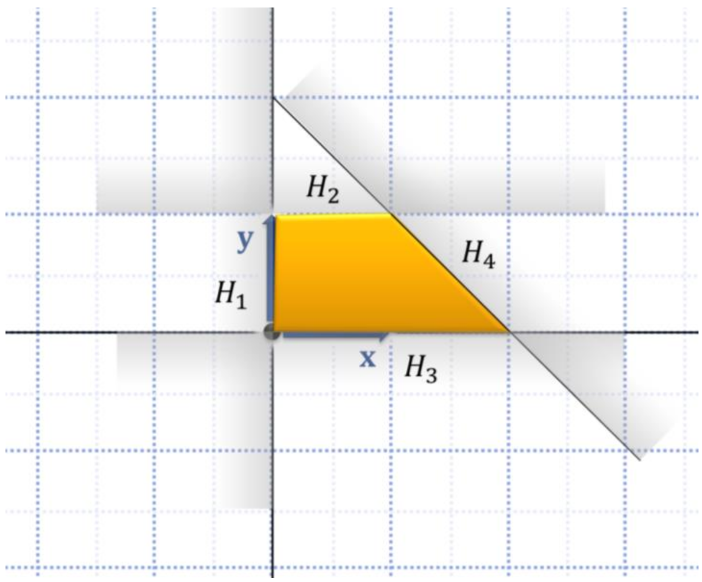


Figure 2: quad from the assignment

$$H_1 \equiv x \geq 0$$
$$H_2 \equiv y \leq 1$$
$$H_3 \equiv y \geq 0$$
$$H_4 : y \leq 2 - x \rightarrow H_4 \equiv x + y \leq 2$$

b) Based on this description, can you devise a test for any point $\mathbf{p} \in \mathbb{R}^2$ to check whether $\mathbf{p}$ is inside the quad or not?

Solution: Sure, just insert the point in equation H_1-H_4 and evaluate all of these; take the logical "and".

c) Imagine you are given the Half-space inequalities H_1, H_2, H_3, H_4 for the four edges of a general quadrilateral. Can you develop an algorithm that tests whether the quadrilateral is convex or not? (triangles are always convex, but quadrilaterals might be both concave and convex, as shown in Fig. 1).

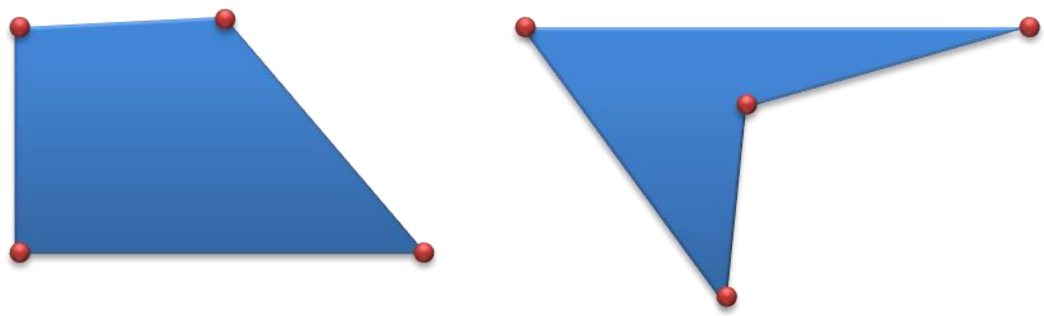


Figure 1: A convex and a concave quadrilateral.

Solution:

Insert point p_i in half-space tests $H_1, \dots, H_4$. For a convex quad, all of the test must be true. A concave quad will return false on some of them.

Remark: In practice, for numerical stability, the test $ax + by + c \leq 0$ should be replaced by $ax + by + c \leq -\epsilon$ for an appropriately small chosen threshold ϵ (maybe $\epsilon = 10^{-4}$ or so).

Assignment #5: *Matrices*

CORE

core topics *important*

This assignment is meant to practice matrix calculations. This is a core skill that is important for the upcoming exam(s).

a) Determine a transformation matrix that rotates by 45° around the origin of the coordinate system. The solution is given below so that you can continue with b),c), but you should build it yourself and briefly explain how you did it.

$$\mathbf{R}_{45^\circ} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

b) Transform the vectors

$$\binom{1}{0}, \binom{0}{1}, \binom{1}{1}$$

by $\mathbf{R}_{45^\circ}$. (Just to practice matrix-vector products.)

Solution:

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

c) Multiply the matrix with itself and show that this is a rotation by 90°.

Solution:

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\cdot\frac{1}{2}\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{R}_{90}$$

d) Show that $(\mathbf{R}_{45^\circ})^8 = \mathbf{I}$.
Hint: Use $\mathbf{A}^{b\cdot c} = (\mathbf{A}^b)^c$ to avoid unnecessary work.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$