
UU Graphics

academic year 2013/14 – 4th period

Theoretical Assignment #2: Linear Algebra II

May 06 2014

Assignment #1: Raytracing! (at least, we will need this later for raytracing...)

We are given the following triangle in \mathbb{R}^3 :

$$T = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid 0 \leq \mu_1, \mu_2 \leq 1, \mu_1 + \mu_2 \leq 1 \right\}$$

In addition, we are given three rays, specified below in their parametric ray equation:

$$R_1 = \left\{ \begin{pmatrix} 0.25 \\ 0.25 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

$$R_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

$$R_3 = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \lambda \in \mathbb{R} \text{ and } \lambda \geq 0 \right\}$$

First, sketch the triangle T and the rays R_1, R_2, R_3 in a sketch/drawing. It is recommended to visualize the situation by projecting on the x-y plane (dropping z or using parallel projection for the z -direction). Which of the three rays R_1, R_2, R_3 intersects triangle T , and why? Can you formally compute the parameters μ_1, μ_2, λ to check your result?

When computing the intersection formally, what kind of system of equations do you need to solve?



Assignment #2: What are these objects?

Please classify the following objects as (i) a linear space, (ii) an affine space, (iii) something different from the two. Explain why.



- a) The set $\{\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors
- b) The set $\{(\lambda_1 + \lambda_3)\mathbf{u} + (\lambda_2 + \lambda_3)\mathbf{v} | \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors
- c) The set $\{\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} + \lambda_1 \lambda_2 (\mathbf{u} + \mathbf{v}) | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors
- d) The set $\{\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} + \mathbf{p} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v}, \mathbf{p} \in \mathbb{R}^3$ are fixed vectors
- e) The set $\{\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} + \mathbf{u} | \lambda_1, \lambda_2 \in \mathbb{R}\}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors

Assignment #3: Practicing Vector Algebra

Proof the following equations by applying the algebraic rules for vector spaces. Instructions: apply algebraic rules to the left-hand side until you get the result on the right-hand side.



- a) First, just compute the expression

$$3 \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) \right) + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

- b) Now show the following: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be vectors, and $\mu, \lambda \in \mathbb{R}$ be real numbers. Then:

$$\lambda(\mathbf{u} + \mu(\mathbf{v} + \mathbf{w})) + 2\mathbf{w} = \lambda\mathbf{u} + \lambda\mu\mathbf{v} + (\lambda\mu + 2)\mathbf{w}$$

In order to prove this result, you can use either the definition of vector spaces from the lecture (as d-tupels) or the axioms listed in Assignment 3 of Tutorial 1 (if you want to go with “pure mathematics”; in this case, cite all of the rules that you are using to convert the left-hand-side into the right-hand one).

- c) Check that (a) yields the same result.
- d) We now practice the sum notation. First, just explain why the following equation holds (by applying the respective rules from Tutorial 1, Assignment 3):

$$(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) \cdot (\mu_1 + \mu_2) = \lambda_1 \mu_1 \mathbf{v}_1 + \lambda_2 \mu_1 \mathbf{v}_2 + \lambda_1 \mu_2 \mathbf{v}_1 + \lambda_2 \mu_2 \mathbf{v}_2$$

- e) Now, explain why this formula is true:

$$\left(\sum_{i=1}^n \lambda_i \mathbf{v}_i \right) \cdot \left(\sum_{i=1}^n \mu_i \right) = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \mu_j \mathbf{v}_i$$



where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $\mu_1, \dots, \mu_k \in \mathbb{R}$, $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^d$.

Assignment #4: Bonus assignment (in reaction to the question in the lecture :-)

We want to build a convex polygon in 2D (we will use a quadrilateral, not a 7-sided Heptagon).



a) Consider a rectangle in \mathbb{R}^2 , spanned by the following four points as corners:

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{p}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \mathbf{p}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{p}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Describe this quadrilateral as intersection of four half spaces: The whole 2D plane can be parametrized by its canonical coordinate system as follows:

$$\mathbb{R}^2 = \left\{ x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$$

Formulate four linear inequalities $H_i(x, y) \equiv [a_1^{(i)}x + a_2^{(i)}y + a_3^{(i)} \leq 0]$ such that their conjunction (applying logical “and”) describes exactly the points inside the quadrilateral.

b) Based on this description, can you devise a test for any point $\mathbf{p} \in \mathbb{R}^2$ to check whether \mathbf{p} is inside the quad or not?

c) Imagine you are given the Half-space inequalities H_1, H_2, H_3, H_4 for the four edges of a general quadrilateral. Can you develop an algorithm that tests whether the quadrilateral is convex or not? (triangles are always convex, but quadrilaterals might be both concave and convex, as shown in Fig. 1).

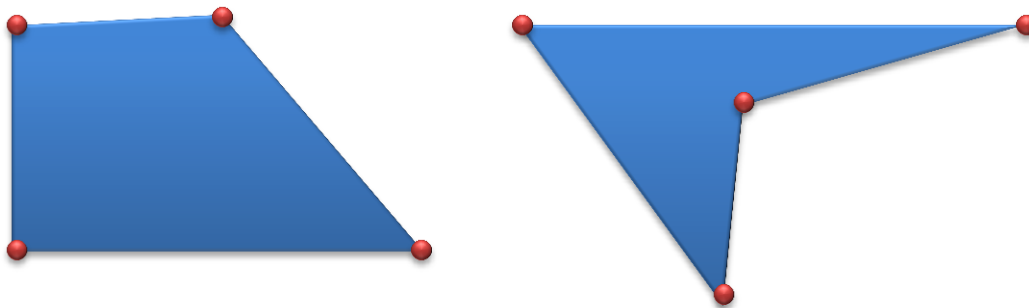


Figure 1: A convex and a concave quadrilateral.

Assignment #5: Matrices

This assignment is meant to practice matrix calculations. This is a core skill that is important for the upcoming exam(s).



- a) Determine a transformation matrix that rotates by 45° around the origin of the coordinate system. The solution is given below so that you can continue with b),c), but you should build it yourself and briefly explain how you did it.

$$\mathbf{R}_{45^\circ} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$$

- b) Transform the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

by \mathbf{R}_{45° . (Just to practice matrix-vector products.)

- c) Multiply the matrix with itself and show that this is a rotation by 90° .
d) Show that $(\mathbf{R}_{45^\circ})^8 = \mathbf{I}$.

Hint: Use $\mathbf{A}^{b \cdot c} = (\mathbf{A}^b)^c$ to avoid unnecessary work.