UU Graphics

academic year 2013/14 – 4th period

Theoretical Assignment #3:
Linear Algebra III – Matrix Inversion and the Scalar Product

May 13 2014

Assignment #1: *Compute the following products*

BASIC
basic topics
study completely

a) Matrix-vector multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 2 - 2 + 3 \\ 8 - 5 + 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

b) Matrix-vector multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(write-out the symbolic solution)

Solution

$$\begin{bmatrix} ax + by + c \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

c) Matrix-matrix multiplication. Compute:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

Solution

just do it.

d) Proof that a uniform scaling matrix $\mathbf{\Lambda}$ always commutes with any other matrix, i.e., $\mathbf{\Lambda} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{\Lambda}$. For simplicity, just consider the 2D case:

$$\underbrace{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\boldsymbol{\Lambda}} \cdot \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\mathbf{M}}$$

Instructions: Compute the matrix-matrix product in both ways and compare. There might be easier proofs, but this assignment is meant to exercise matrix-vector products.

Solution

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix}$$

non-uniform? no

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} \\ \lambda_2 a_{21} & \lambda_2 a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} \\ \lambda_1 a_{21} & \lambda_2 a_{22} \end{bmatrix}$$

Assignment #2: *Matrix Inversion*

CORE
core topics
important

a) Consider the following linear system of equations:

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

Compute x, y, z using Gaussian elimination!

Solution

$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

$$z = -1$$
$$\frac{1}{2}y = 1 + \frac{1}{2} \rightarrow y = 2 + 1 = 3$$
$$2x + 3 + 1 = 8 \rightarrow x = 2$$

Remark: The same example is also discussed at http://en.wikipedia.org/wiki/Gaussian_elimination.

b) Compute the inverse of the following matrix using Gaussian elimination:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Solution

$$\left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & \frac{1}{3} & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\frac{4}{9} & -\frac{4}{3} \end{array}\right)$$

$$\rightarrow \cdots$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{2} & -\frac{3}{4} \end{array}\right)$$

Assignment #3: *Orthogonalization*

BASIC
basic topics
study completely

a) Consider the following three vectors:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Convert the three vectors into an orthogonal coordinate system using the Gram-Schmidt Algorithm. Specifically, consider $\mathbf{x}_1$ fixed and modify $\mathbf{x}_2$ such that it is orthonormal to $\mathbf{x}_1$. Then modify $\mathbf{x}_3$ such that it is orthogonal to $\mathbf{x}_1, \mathbf{x}_2$. **Hint:** Check first, if pairs of vectors are already orthogonal to save some work!

Solution

$\mathbf{x}_1$ and $\mathbf{x}_2$ are already orthogonal. So we can continue with $\mathbf{x}_3$: $\mathbf{x}_3$ is orthogonal to $\mathbf{x}_2$; we only need to orthogonalize with $\mathbf{x}_1$:

$$\mathbf{x}_3' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\sqrt{2}^{-1} \right) \cdot \begin{bmatrix} \sqrt{2}^{-1} \\ 0 \\ \sqrt{2}^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ \text{-}1/2 \end{bmatrix}$$

b) Consider the vector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Build an orthogonal basis of $\mathbb{R}^3$ that contains $\mathbf{x}_1$ as one basis vector. (Now) use the cross-product for this (a solution with the scalar product has been obtained in (a)). **Hint:** See textbook chapter 2.4.6.

Solution

Compute for example

$$\mathbf{x}_2 = \mathbf{x}_1 \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2$$

Assignment #4: *Implicit modeling*

Geometric objects can be modeled by *parametric equations* or by *implicit equations*. We have seen parametric equations already in the lecture. For example, a line in $\mathbb{R}^2$ can be represented parametrically as

$$\mathbf{x}(t) = \mathbf{p} + t \cdot \mathbf{r}, \;\; \mathbf{p}, \mathbf{r} \in \mathbb{R}^2, t \in \mathbb{R} \qquad (1)$$

The scalar *parameter* t allows us to move along the line. This is the reason for calling the format parametric – the describe primitive (the line) can be scanned by varying the parameter.

In contrast, implicit equations express only a logical condition that allows us to test whether we are on the object or not. For example, an implicit line equation will look like that:

$$\text{for all } \mathbf{x} \text{ on the line: } \mathbf{x} \cdot \mathbf{n} = d, \;\; \mathbf{n}, \mathbf{x} \in \mathbb{R}^2, d \in \mathbb{R}, \mathbf{n} \neq \mathbf{0} \qquad (2)$$

This equation also describes a line in $\mathbb{R}^2$, but it does not give us a parameter to walk along the line. On the other hand, we can directly test whether some point $\mathbf{x}$ is located on the line without solving a system of equations. Further, the implicit form is often easer to setup for complicated shapes.

With these remarks on the background, solve the following assign to understand the concept better.

a) Consider a line through the origin. How will equations (1) and (2) above (always) look like in this case?

Solution:

p = 0, *d* = 0

b) What is the geometric meaning of the vectors $\mathbf{n}$ and $\mathbf{r}$ in equation (1) and (2), respectively? Similarly, what is the meaning of point $\mathbf{p}$ and the scalar d? Hint: In case you do not see the relation, try a few examples and draw the situation (including all of the points and vectors).

Solution:

$\mathbf{n}$ is a vector that is normal to the line. $\mathbf{r}$ is a vector that is tangential to the line (direction vector).

$\mathbf{p}$ is the offset vector and d is a distance vector projected on the normal.

c) Convert between the representations of Eq. (1) and Eq. (2). What is the general rule here? **Hint:** This assignment is easy after answering (b).

Solution:

For example: $\mathbf{n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{r}$; other direction: $\mathbf{r} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{n}$

$\mathbf{d} = \mathbf{n} \cdot \mathbf{p}$; other direction: $\mathbf{p} = \frac{d}{\|\mathbf{n}\|^2}\mathbf{n}$.

d) Does the implicit representation of a line still work in $\mathbb{R}^3$? Hint – no, but what does such an equation describe here? Could you think of a way to fix this (i.e., come up with a modified, implicit formulation that still describes a line)? **Hint:** Use more than one equation.

Solution:

Of course not – the equation describes a plane. We could still retain an implicit line representation by using the intersection of two planes (e.g., constructing to orthogonal normal vectors using the method from 3b).

Assignment #5: *Non-linear objects*

Hint: Work on Assignment 4 before this one.

a) Use the scalar product to create an equation that describes all point on a unit circle in the plane around the origin (i.e., radius = 1, center = $\mathbf{0}$).

Solution:

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = 1 \rightarrow \mathbf{x} \cdot \mathbf{x} = 1 \text{ (short: } \mathbf{x}^2 = 1)$$

b) Create the same equation for an arbitrary circle with any radius $r \geq 0$ and arbitrary center $\mathbf{c} \in \mathbb{R}^2$.

Solution:

$$(\mathbf{x} - \mathbf{c})^2 = r^2$$

c) Develop a formula for intersecting the general circle with a line. Hint: Express the line in parametric form and compute the intersection.

Solution:

$$\begin{aligned}
&(\mathbf{x} - \mathbf{c})^2 = r^2 \wedge \mathbf{x} = t\mathbf{r} + \mathbf{p} \\
&\rightarrow (t\mathbf{r} + \mathbf{p} - \mathbf{c})^2 = r^2 \\
&\rightarrow t^2\mathbf{r}^2 + t(\mathbf{p} - \mathbf{c}) \cdot \mathbf{r} - r^2 + (\mathbf{p} - \mathbf{c})^2 = 0
\end{aligned}$$

Solving the quadratic equation in t:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \begin{aligned} a &= \mathbf{r}^2, \\ b &= (\mathbf{p} - \mathbf{c}) \cdot \mathbf{r}, \\ c &= (\mathbf{p} - \mathbf{c})^2 - r^2 \end{aligned}$$

Assignment #6: *Solution spaces of linear systems*

We are given a linear system of equations

$$\mathbf{M} \cdot \mathbf{x} = \mathbf{y}, \quad \mathbf{M} \in \mathbb{R}^{n \times m}, \mathbf{x} \in \mathbb{R}^{n}, \mathbf{y} \in \mathbb{R}^{m}$$

The matrix $\mathbf{M}$ is a general matrix with $\mathbf{n}$ columns and $\mathbf{m}$ rows; it is not necessarily invertible. This means that the system might have no solution at all, exactly one solution, or even more than one solution. We now want to understand the structure of the solution space.

a) Show that the system has a solution if and only if $\mathbf{y} \in \operatorname{span}(\mathbf{M})$.

Solution

First: "$\Rightarrow$"

$$\mathbf{y} \in \operatorname{span}(\mathbf{M}) \Rightarrow \exists x_1, \dots, x_n : \mathbf{y} = \sum_{i=1}^{n} x_i \, \mathbf{m}_i \text{ (columns of } \mathbf{M}) = \mathbf{M}\mathbf{x}$$

Then: "$\Leftarrow$" read the argument above backwards.

b) Let $\mathbf{x}_1, \dots, \mathbf{x}_l$ be solutions of the system of equations. Then, show that the vector

$$\mathbf{x}' = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_l \mathbf{x}_l$$

is also a solution for any coefficients $\lambda_1, \dots, \lambda_l \in \mathbb{R}$ with

$$\lambda_1 + \cdots + \lambda_l = 1$$

This type of constrained linearly combination (weights sum to one) is also called an *affine* combination. Any affine subspace is closed under affine combinations (but not general linear combinations – can you give a counter example?). In a more general framework, the property above can also be used to characterize the affine spaces.

Solution:

$$\mathbf{x}' = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_l \mathbf{x}_l, \text{ with } \mu_1 + \cdots + \mu_l = 1 \text{ for solutions } \mathbf{x}_1, \dots, \mathbf{x}_l$$

We know that

$$\mathbf{M} \cdot \mathbf{x}_i = \mathbf{y}$$

Hence,

$$\mathbf{M} \cdot \mathbf{x}' = \mathbf{M} \cdot (\lambda_1 \mathbf{x}_1 + \cdots + \lambda_l \mathbf{x}_l) = \lambda_1 \mathbf{M} \cdot \mathbf{x}_1 + \cdots + \lambda_l \mathbf{M} \cdot \mathbf{x}_l = \lambda_1 \mathbf{y} + \cdots + \lambda_l \mathbf{y} = 1 \cdot \mathbf{y} = \mathbf{y}$$

Remark: What we have learned here is that the solution space of a linear system of equations is either empty or an affine subspace of $\mathbb{R}^n$. This means, it is the empty set, a single point, a line, a plane, or a *d*-dimensional hyperplane with *d* up to *n* (i.e., the whole $\mathbb{R}^n$).