UU Graphics

academic year 2013/14 – 4th period

Additional Practice Assignments
Lecture Summary / Final Exam preparation

June 24 2014

This assignment sheet is intended to serve two purposes:

- To further practice some analytical / theoretical skills.
  (In response to various feedback that we got.)
- To help you prepare for the final exam.
  (In place of a practice exam.)

Accordingly, the sheet provides a mixture of assignments. In particular, assignment 2 was based on student feedback. However, all assignments are good preparations for the exam (including the assignment 2) and could be relevant for the final exam (as well as all of the earlier tutorials, T1-T6).

Important remarks:

- This is not a practice exam! In particular:
- The level of difficulty might or might not be representative (the final exam could also be harder or much easier than the assignments shown here).
- The time needed for working through these assignments is different from the final exam.
- The exam might also cover different topics than the ones shown here; make sure to look at all topics of the lecture for your preparations.
- **The exam questions will look different.** The assignments here are formulated in a way such that you think about the topic and learn as much as possible, not for having clear and concise answers (as required for exam questions).
- Our intention at this point is to help you to further develop your skills, which will certainly be beneficial for the exam, but you cannot rely on this selection being comprehensive or fully representative for the exam.

Treat this more as an additional tutorial sheet, created with the intention of preparing you for the exam, but not as a replacement for studying all of the past lectures and tutorials. Nonetheless, it should be quite useful to work on these assignments.

Additional Tutorial session: There will be again three parallel tutorial session on Thursdays (June 26, after the lecture, as always). The tutors there will help you with these assignments, and can also answer questions concerning the previous topics. Do not miss this opportunity for additional personal coaching and feedback!

Good luck with your final preparations!

Assignment #1: *Homogeneous coordinates*

Consider 3D vectors represented as 4D homogeneous coordinates:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, \text{ represented as } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4.$$

(a) The homogeneous representation distinguishes between Euclidean points and their homogeneous coordinates (representations). Please explain (in one sentence or formula), which homogeneous coordinates are representing the same Euclidean point.

(b) Which Euclidean points are represented by the following homogeneous coordinates?

$$\underset{\text{(i)}}{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}} \quad \underset{\text{(ii)}}{\begin{pmatrix} 2 \\ 3 \\ 6 \\ 1 \end{pmatrix}} \quad \underset{\text{(iii)}}{\begin{pmatrix} 6 \\ 4 \\ 2 \\ 2 \end{pmatrix}} \quad \underset{\text{(iv)}}{\begin{pmatrix} 4 \\ 0 \\ 4 \\ 4 \end{pmatrix}} \quad \underset{\text{(v)}}{\begin{pmatrix} 1 \\ 1.5 \\ 3 \\ 0.5 \end{pmatrix}} \quad \underset{\text{(vi)}}{\begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}} \quad \underset{\text{(vii)}}{\begin{pmatrix} -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}} \quad \underset{\text{(viii)}}{\begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}}$$

(d) A rather theoretical assignment, but not difficult at all: As introduce in the lecture, a *projective map* is a linear map in homogenous coordinates (a 4×4 matrix in our case here). Proof the following: any non-zero multiple of such a matrix (i.e., we multiply all entries simultaneously by the same non-zero number) has the same effect on the result if it is interpreted as Euclidean point.

Hint: show that

$$\forall \lambda \neq 0: \quad (\lambda \cdot \mathbf{M}) \cdot \mathbf{x} \equiv \mathbf{M} \cdot \mathbf{x}\,,$$

where $\lambda \cdot \mathbf{M}$ denotes the matrix where all entries of $\mathbf{M}$ have been multiplied by λ and $\equiv$ denotes equality with respect to the Euclidean point being represented.

Assignment #2: *Homogeneous transformations [feedback]*

We now consider homogeneous transformations (projective maps) of 2D Euclidean points (3D homogeneous coordinates). Our goal is to derive various transformation matrices.

(a) Provide a homogeneous 3×3 transformation matrix that translates a 2D point $\mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ (specified in homogeneous coordinates) by a vector $\mathbf{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$. What happens if you input a general point $\mathbf{x}' = \begin{pmatrix} wx \\ wy \\ w \end{pmatrix}$ to the same mapping – is the result still consistent?

(b) Write down a homogenous 3×3 transformation matrix that represents a rotation around the origin by angle α.

(c) Now, by combining the two results, describe how we can compute a single homogeneous 3×3 transformation matrix that rotates points around an arbitrary rotation center $\mathbf{p}$ in the plane.

(d) Alright – let's move on to something more difficult. Our task is to create a homogenous transformation matrix that meets the following conditions:

- Again, the 3×3 matrix should operate on 2D points (when interpreted as Euclidean points), represented in 3D homogenous coordinates.
- We fix the following conditions:
  - The origin, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, should be mapped to a 2D point $\mathbf{o}' = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$ (i.e., $\begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix}$ in homogenous coordinates).
  - The point $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, should be mapped to a 2D point $\mathbf{u}' = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$ (i.e., $\begin{pmatrix} u_x \\ u_y \\ 1 \end{pmatrix}$ in homogenous coordinates).
  - The point $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, should be mapped to a 2D point $\mathbf{v}' = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ (i.e., $\begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}$ in homogenous coordinates).

These conditions (mapping the origin and the coordinate axes) fix a (unique) affine map. Compute this affine map and represent it in homogeneous coordinates.

(e) Now, one more step: We do not map the coordinate system, but we allow a general reference frame to be mapped. This is the same that we did in the lecture for deriving the mapping for texture mapping; now we want to express this in homogeneous coordinates.

Again, we want to meet the following conditions

- Once more, our 3×3 matrix should operate on 2D points (when interpreted as Euclidean points), represented in 3D homogenous coordinates.
- We fix the following additional conditions: We are given three source points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, not co-linear (not on a single line) and map them to points $\mathbf{p}_1', \mathbf{p}_2', \mathbf{p}_3'$. All of these are Euclidean 2D points (3D in homogeneous coordinates).

These conditions (mapping the origin and the coordinate axes) again fix a (unique) affine map. Compute this affine map and represent the mapping in homogeneous coordinates.

(f) With this formula at hand, can you solve the view-port transformation problem? We are given normalized device coordinates ($[-1,1]^2$, origin in the middle, x-axis pointing to the right, y-axis pointing upward) and want to map to screen coordinates (origin upper left corner, $w \times h$ pixels, y-axis pointing downwards).

(g) Bonus assignment (more difficult!): The conditions in (e) fix a unique affine map, but not a unique projective map (3×3 homogeneous transformation). Why is this the case? (Why is the affine map uniquely defined, but the homogeneous one not yet?). Further, show that fixing 4 points (except from degenerate cases, such as collinear source points) fixes a projective map (when considering its action on Euclidean points; such a map is called a *homography*). Remark: the last part of this question is beyond our lecture; the "4-point" question will not show up in any exam. But it is still interesting in order to understand this better!

ADV
advanced topics
main ideas

Assignment #3: *Raytracing*

Some raytracing questions: First, complexity.

(a) What is the runtime complexity of naïve raytracing (no acceleration data structures) with respect to the following to complexity parameters: n triangles, m pixels.

(b) How does it compare to z-Buffering (also a naïve implementation without any culling / multi-resolution, or the similar)?

(c) Can you construct a worst case scene with n triangles for which (no matter how large we prescribe n to be) the asymptotic complexity of both algorithms is the same? How realistic is this case (should we worry about it in practice)?

Further, geometric ray-tracing algorithms are also important!

(d) You are given a ray $\mathbf{x}(\mu) = \mathbf{p} + \mu\mathbf{t}$, that hits a surface in point $\mathbf{x}_0$ with normal $\mathbf{n}$. Compute the reflected ray. You can assume that $\mathbf{n}$ and $\mathbf{t}$ are normalized.

(e) Compute the intersection of a ray and a plane: Assume you are given a ray $\mathbf{x}(\mu) = \mu\mathbf{t} + \mathbf{p}$ and a plane $\mathbf{x}(\lambda_1, \lambda_2) = \mathbf{p}_2 + \lambda_1\mathbf{t}_1 + \lambda_2\mathbf{t}_2$. Describe how you can compute the intersection point of ray and plane, or detect that no single intersection point exists. As computational primitives, you are allowed to solve linear systems (this procedure will return an error code if no solution is found).

Finally, put the different algorithms into perspective:

(f) Compare the shadow-mapping technique (i.e., computing a depth texture by z-Buffer rendering from the light source, then rendering with shadows in a second pass) with ray-traced shadows. What are the main advantages and disadvantages of the techniques? Write a short discussion that mentions 2-3 advantages of one technique over the other (preferably, for either one).

Assignment #4: *Perspective*

We now consider the projection matrix for perspective projection. We are given the following specification

- The screen has $w \times h$ square pixels.
- The vertical viewing angle is α.
- The camera coordinate system is the simplest one:
    - The projection center is in the origin.
    - The camera looks down the z-axis (viewing direction); the image plane is perpendicular to the z-axis.
    - The u- and v-axis of the camera coordinates are the x- and y-axis of the world coordinates (so no change here).
- In short: this is the exact same situation that we considered in the lecture for deriving the projection matrix.

(a) Write down a perspective projection matrix in homogeneous coordinates (4×4). The output for the z-coordinate can be arbitrary.

(b) Use the intercept theorem to explain why this matrix is correct.

Assignment #5: *Miscellaneous*

core topics *important*

Some additional questions concerning various topics.

(a) Shading: Describe the difference between normal interpolation (“Phong-shading”) and color-interpolation (“Gouraud-shading”). Name one advantage for each method over the other.

(b) Local illumination models: Write down the equation for the specular highlight in the Phong illumination model (not to be mixed up with Phong shading as name for normal interpolation, as discussed above). Describe in 1-2 sentences how the model works.

(c) You are given triangles in counter-clockwise (CCW) vertex order, i.e., when viewed from the outside, the vertices are ordered CCW. You also now the viewing direction ($\mathbf{w}$-vector of the camera coordinates system). Describe how you can implement backface culling using difference vectors between vertices and the cross-product and dot product.

(d) You are given a 2×2 matrix with orthonormal column vectors (a so-called orthogonal matrix). Proof that the transpose of this matrix is its inverse. **Hint:** Write the product of the matrix with its transpose as multiple scalar products.

(e) Consider the simple painter’s algorithm (just sorting triangles in depth order). Sketch an example scene for which this algorithm cannot work correctly, no matter how we sort the triangles (i.e., we must cut triangles into pieces in order to get the correct solution). Explain your sketch briefly.