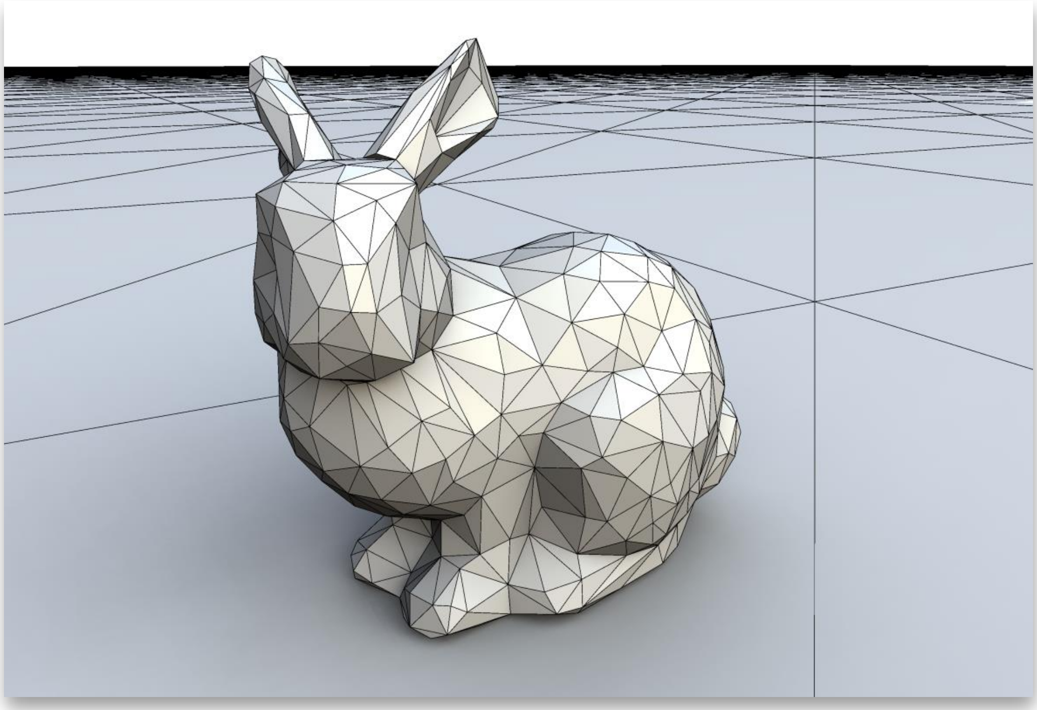
Graphics 2014



Linear Algebra II

Linear Maps & Matrices

[Faculty of Science]
Information and
Computing Sciences

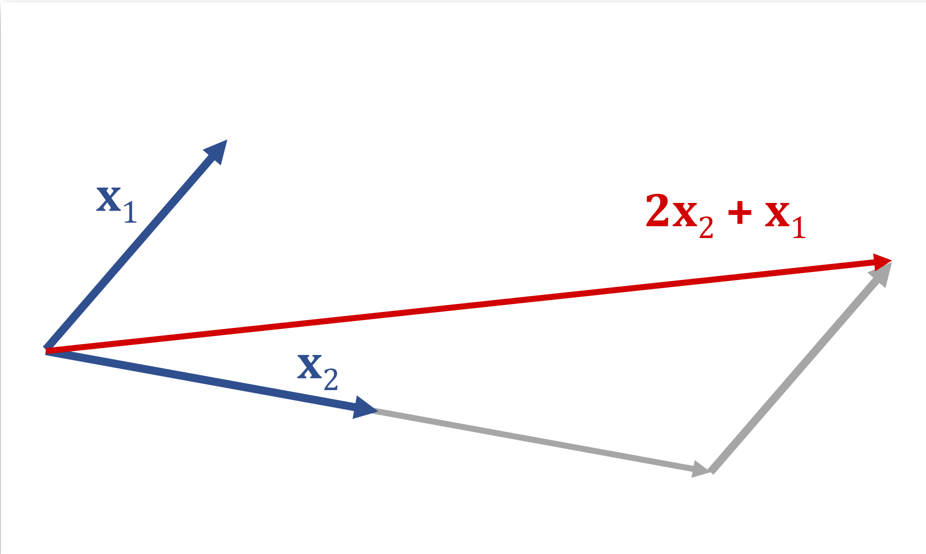
Universiteit Utrecht

Linear Maps & Matrices

CORE

core topics
important

Linear Combinations



Linear Combinations

$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$$

Algebra

Linear Combinations as Mappings

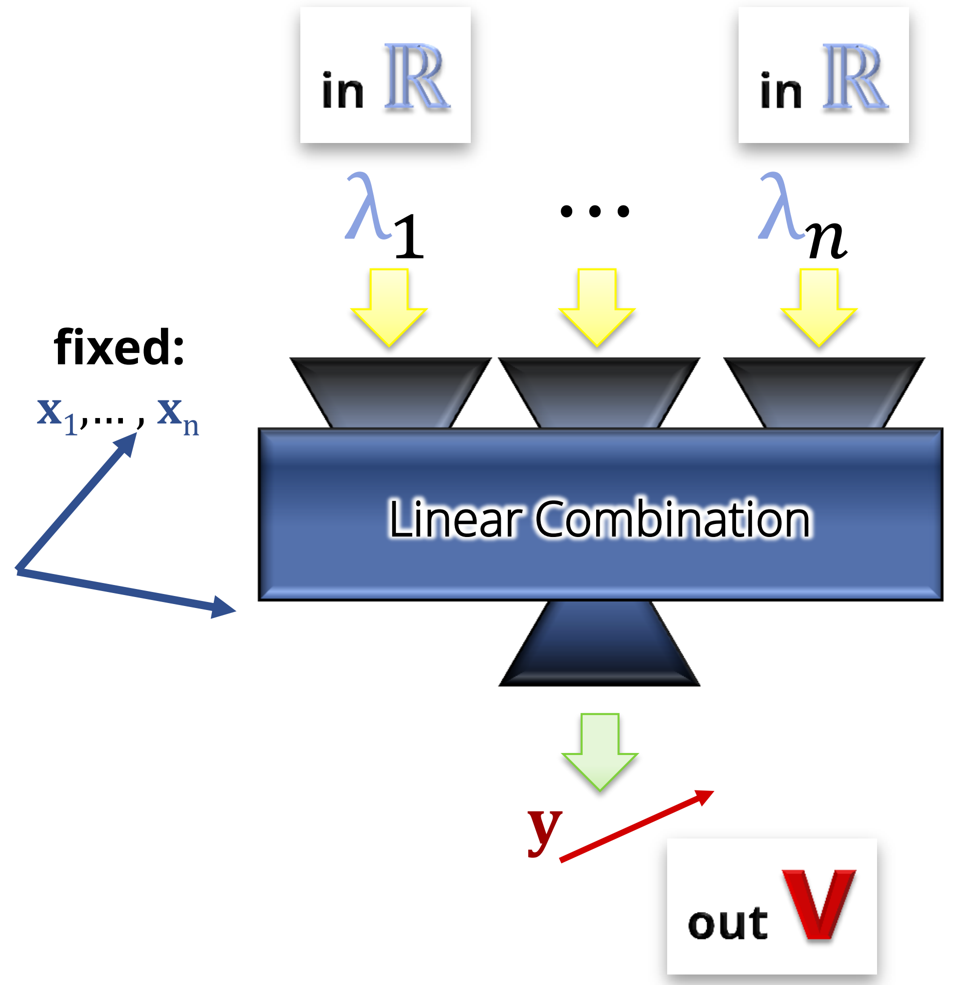
- Fix vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$.
- Factors $\lambda_1, \ldots, \lambda_n \ \rightarrow \mathbf{y}$

Linear Mappings

Linear Map

- Fix vectors
  $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$
- Input coordinates
  $\lambda_1, \ldots, \lambda_n$
- Output vector
  $\mathbf{y} \in \mathbb{R}^m$

$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$$



Map $\lambda_1, \ldots, \lambda_n \to \mathbf{y}$ is called a *linear map*

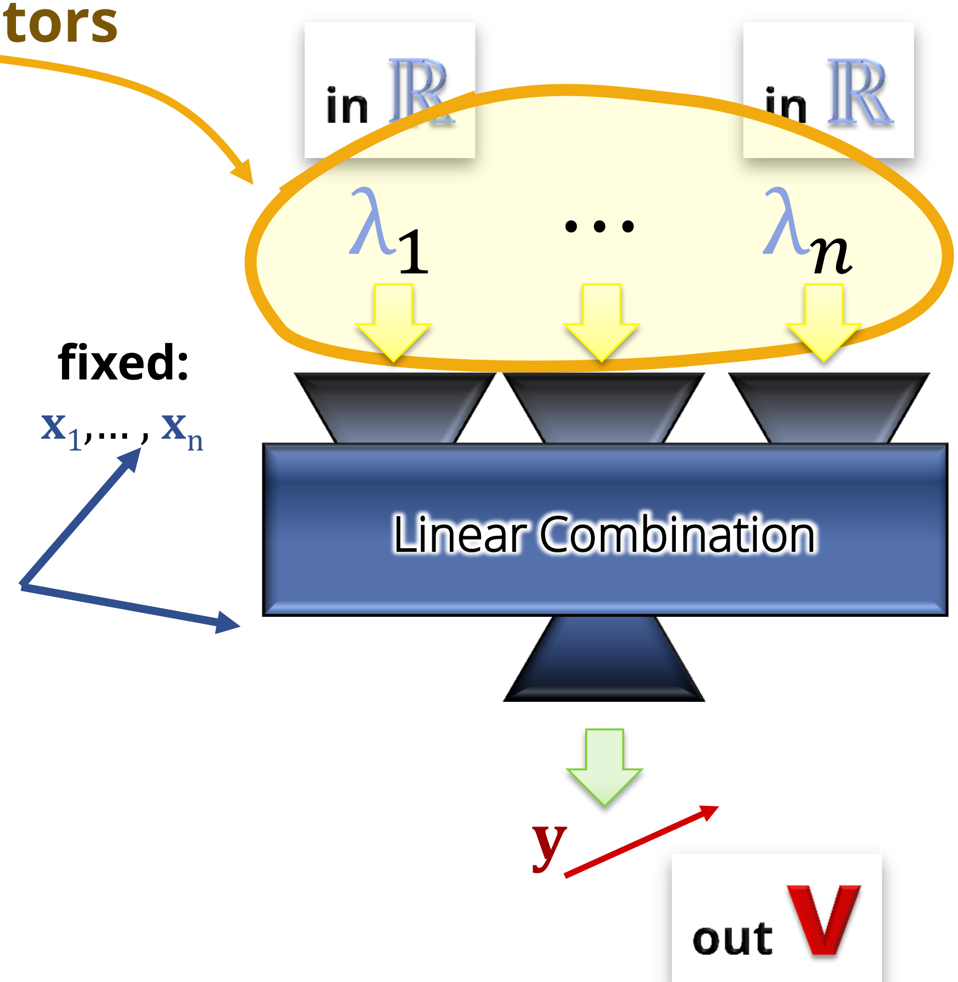
Linear Mappings

Linear Map

Input Vectors

- Fix vectors
  $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$
- Input coordinates
  $\lambda_1, \ldots, \lambda_n$
- Output vector
  $\mathbf{y} \in \mathbb{R}^m$

$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$$



Map $\lambda_1, \ldots, \lambda_n \to \mathbf{y}$ is called a *linear map*

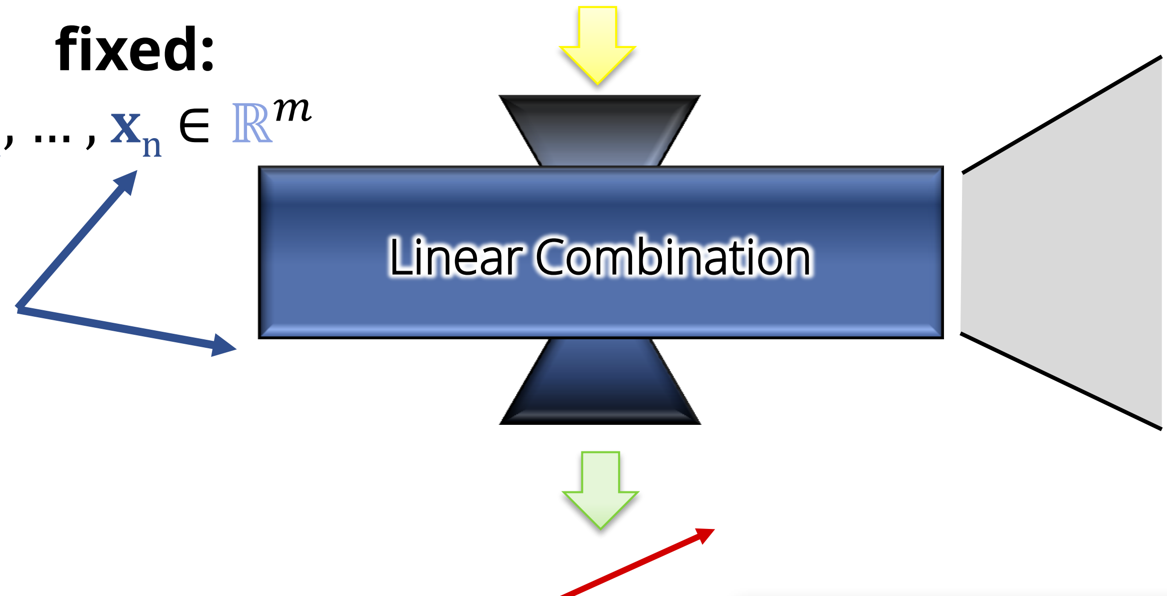
Linear Mappings

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

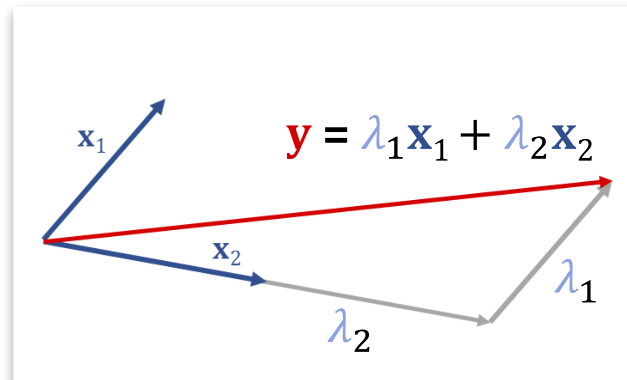
$$\mathbf{V}_{in} = \mathbb{R}^n$$

fixed:

$\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$



How the machine works



$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$$\mathbf{V}_{out} = \mathbb{R}^m$$

Matrix Representation

$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$$

Matrix Representation

$$\mathbf{y} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$$

$$= \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{m,i} \end{bmatrix}$$

$$= \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

Short

$$\mathbf{y} = \mathbf{X} \cdot \boldsymbol{\lambda}$$

Matrix

$$\mathbf{X} = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}$$

Vectors

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Convention

n $columns$

m $rows$

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix}$$

Taken from Textbook [Shirley et al.]

- Matrix elements

$$x_{row,column}$$

- Row first, then column
  - "*y*"-coordinate of the array first (unintuitive, but common convention)

Matrix Representation

Matrix-vector product

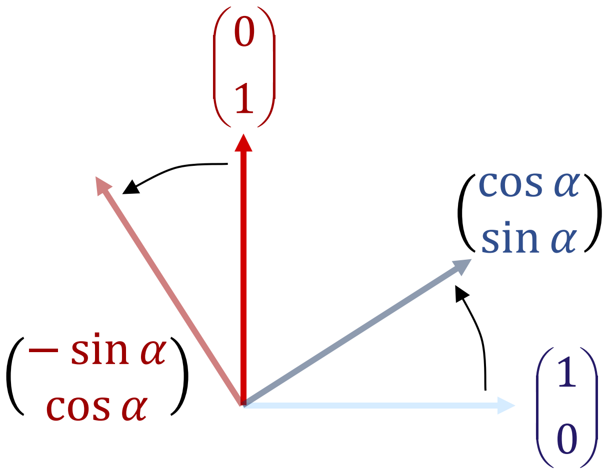
$$\mathbf{y}(\boldsymbol{\lambda}) = \begin{bmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_n \\ | & & | \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

Construction

- Maps from $\mathbb{R}^n \to \mathbb{R}^m$
  - $\boldsymbol{\lambda} \in \mathbb{R}^n$
  - $\mathbf{x}_i \in \mathbb{R}^m \quad \Rightarrow \quad \mathbf{y} \in \mathbb{R}^m$
- Columns of $\mathbf{X}$ = images of the basis vectors of $\mathbb{R}^n$

Example

Example: rotation matrix

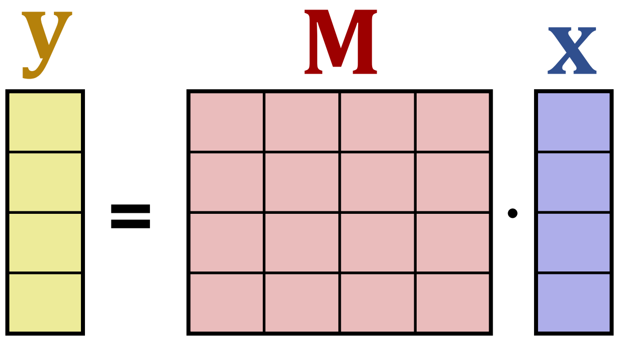


$$\mathbf{M}_{rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

General Matrix Product (Notation)

Algebraic rule:

- Vector-matrix product:



$$\mathbf{M} \cdot \mathbf{x} = \mathbf{y}$$

General Matrix Product (Notation)

Algebraic rule:

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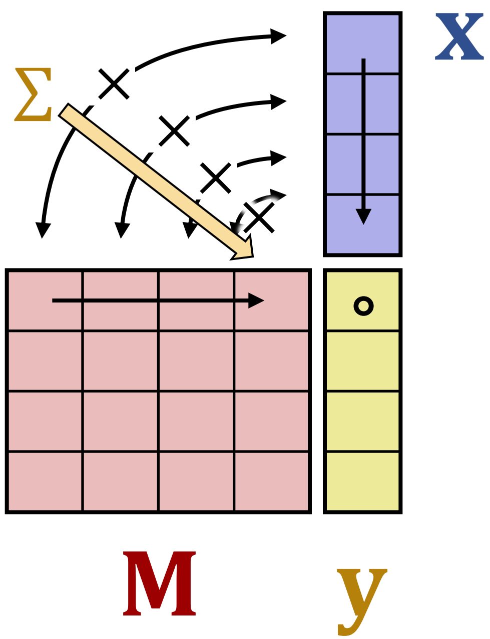

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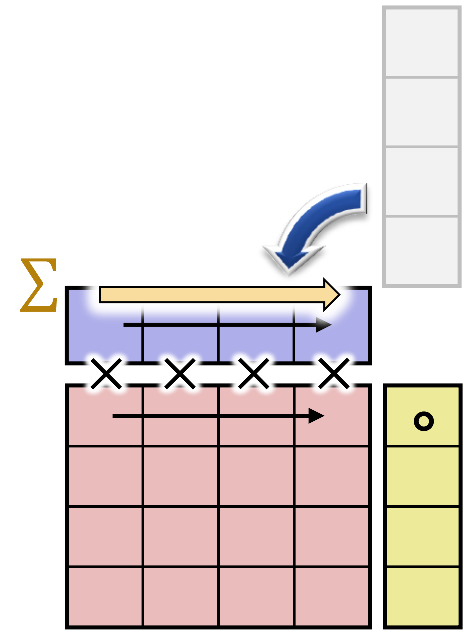

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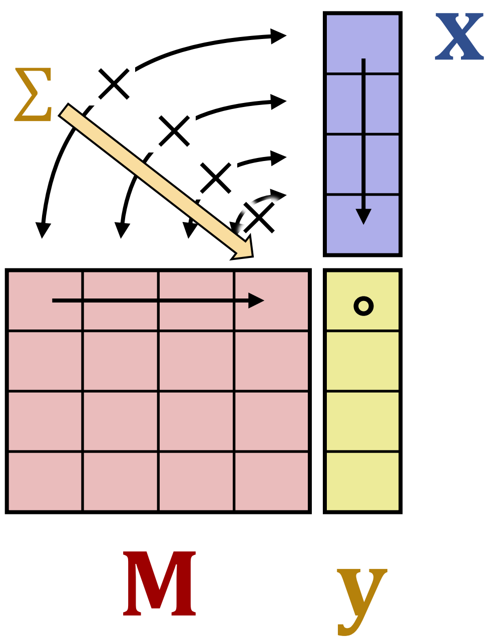

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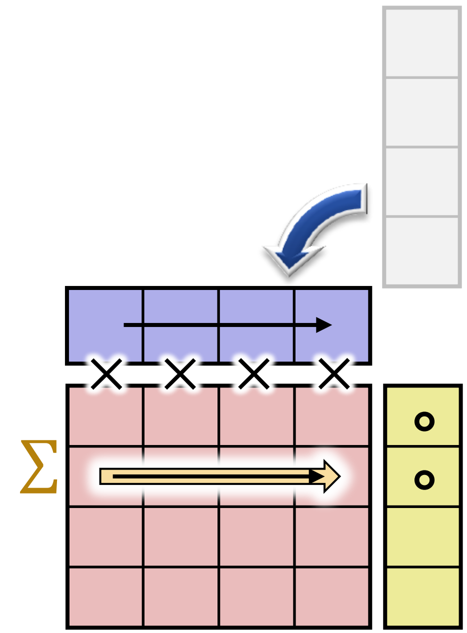

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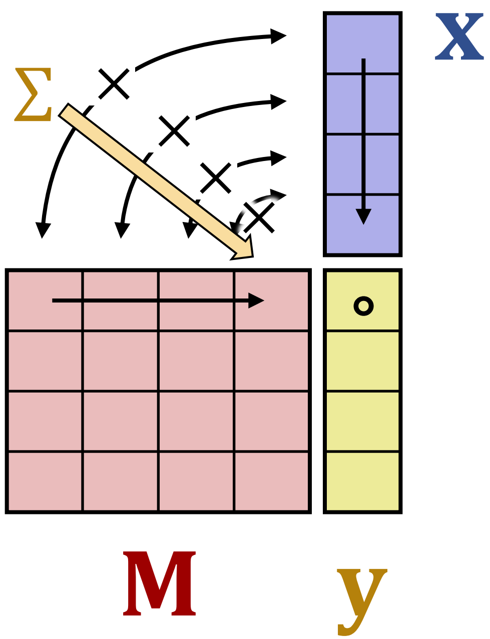

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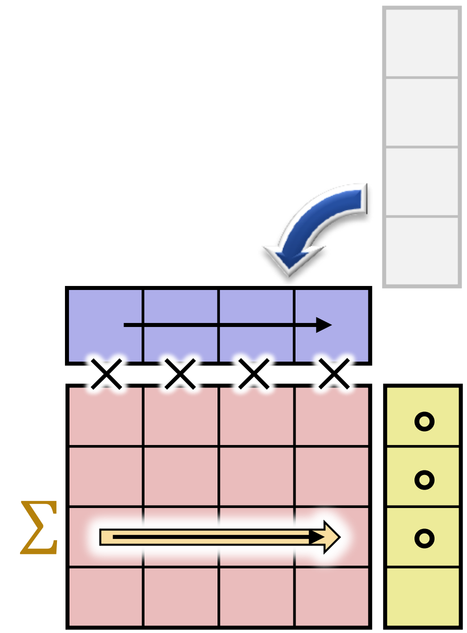

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Algebraic rule:

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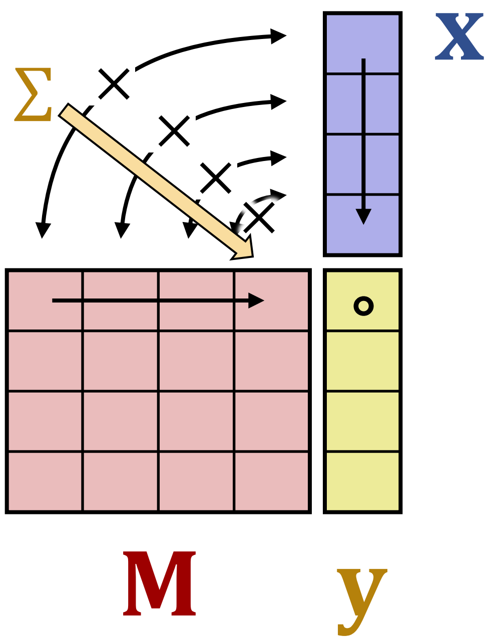

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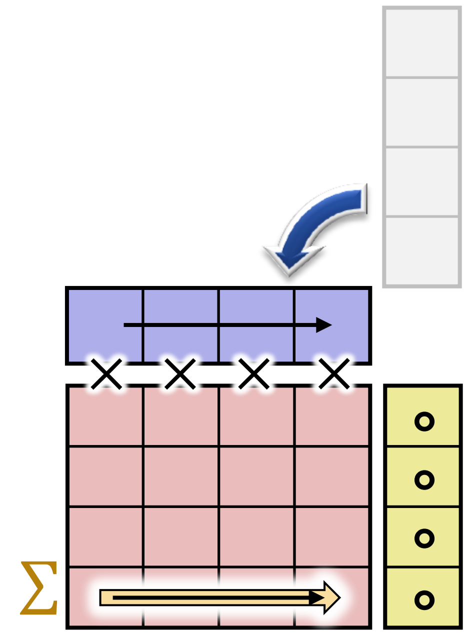
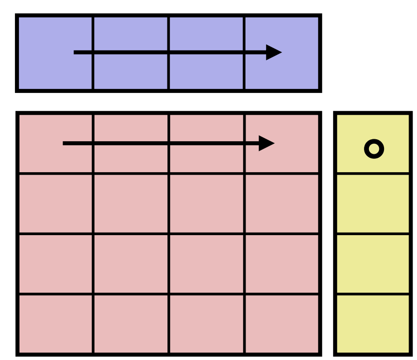


Matrix Representation

Matrix-Vector Multiplication

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,1} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} := \sum_{i=1}^{n} \lambda_i \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{m,i} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \cdot x_{1,1} + \cdots + \lambda_n \cdot x_{1,n} \\ \vdots \\ \lambda_1 \cdot x_{m,1} + \cdots + \lambda_n \cdot x_{m,n} \end{bmatrix}$$



Standard Transformations

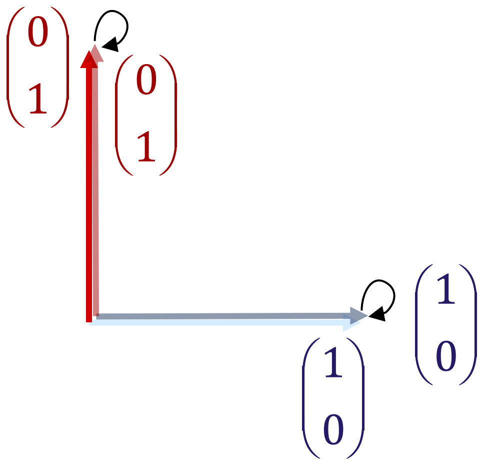
BASIC

basic topics

study completely

Identity Transform

Example: identity matrix

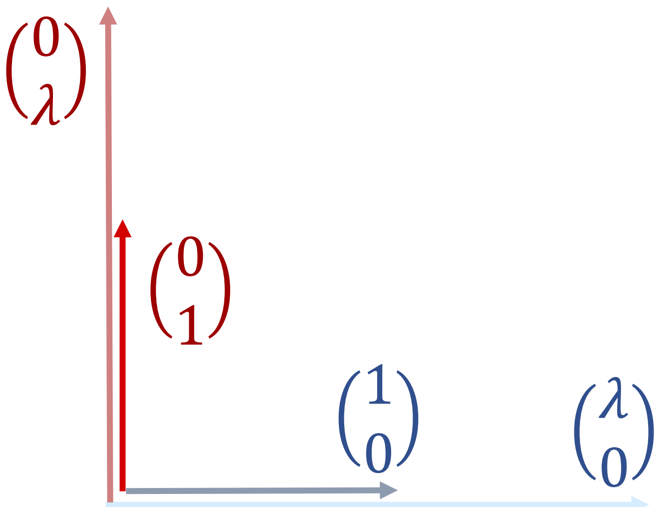


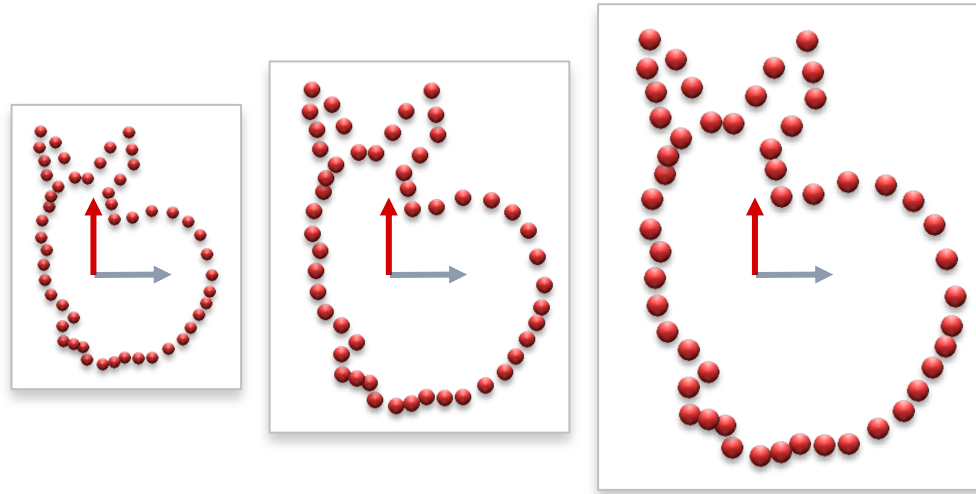
$$\mathbf{M}_{identity} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

General case

$$\mathbf{I}: \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Scaling (Center = Origin)

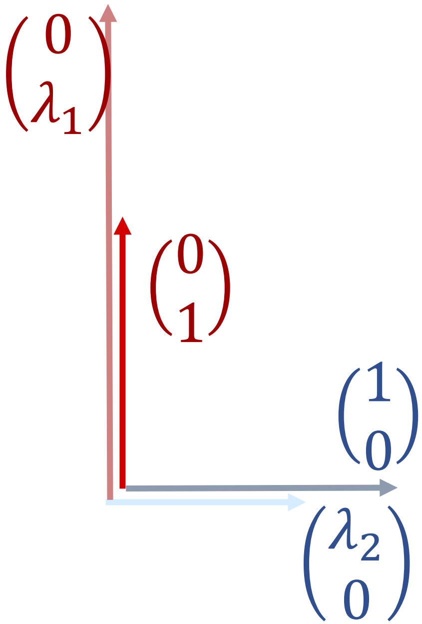


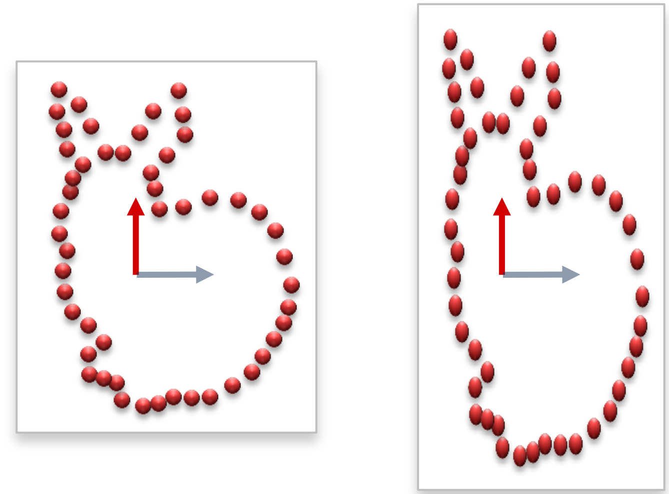


General case

$$\mathbf{S}_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{S}_\lambda = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Non-Uniform Scaling

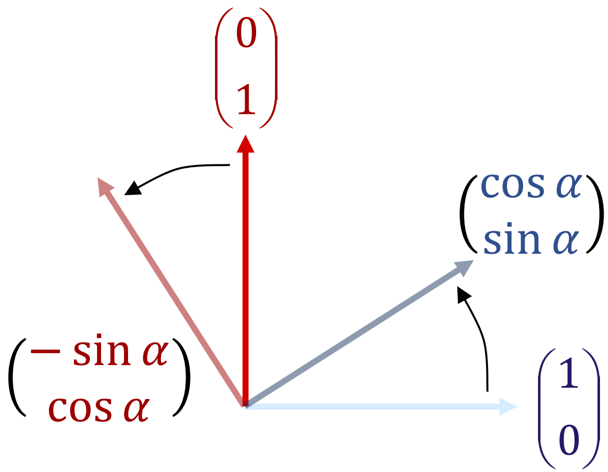


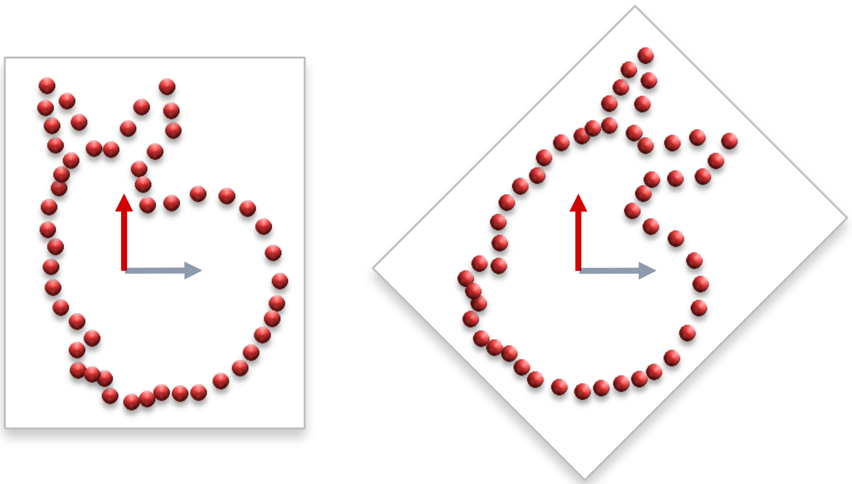


General case

$$\mathbf{S}_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n, \qquad \mathbf{S}_\lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_3 \end{bmatrix}$$

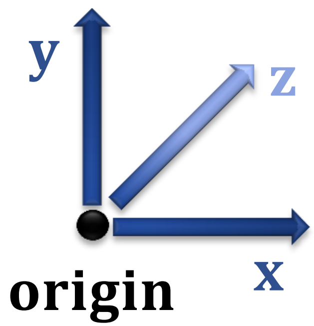
Rotation (2D)





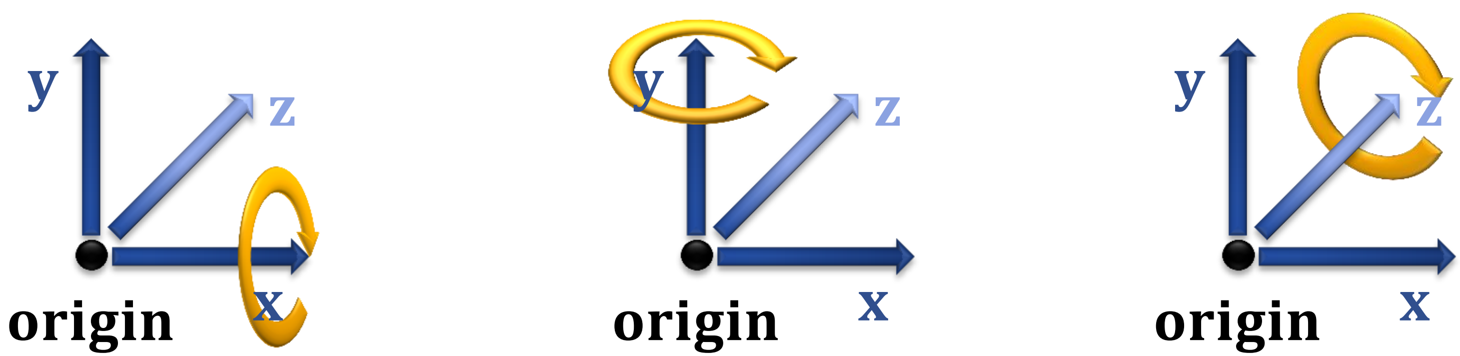
$$\mathbf{M}_{rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Rotation (3D)

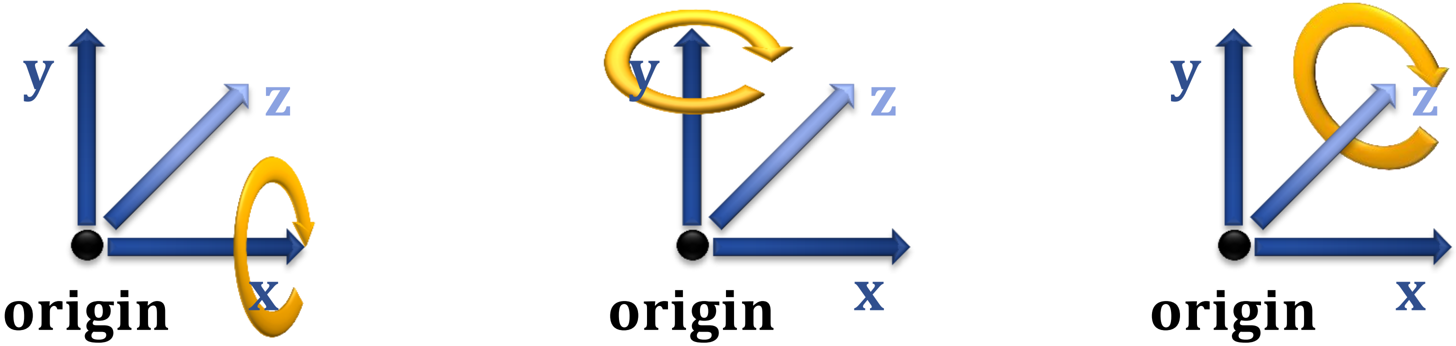


$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$V = \mathbb{R}^3$$



Rotation (3D)

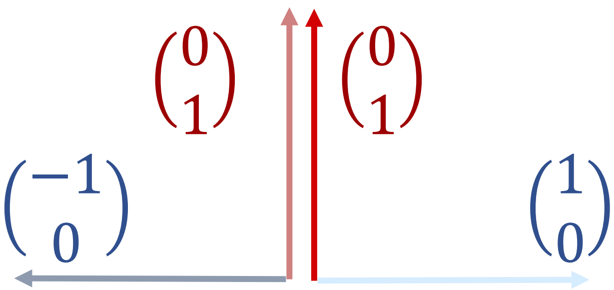


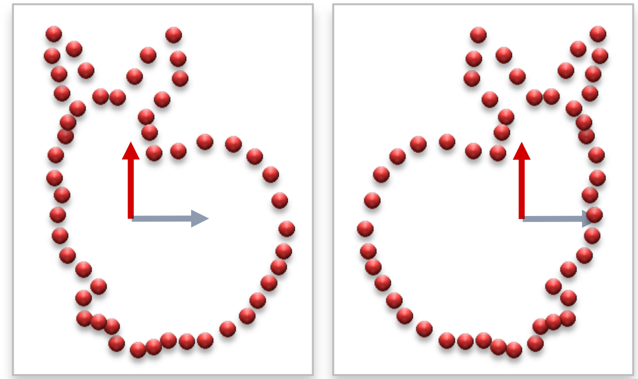
$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\mathbf{R}_z = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_y = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Reflection





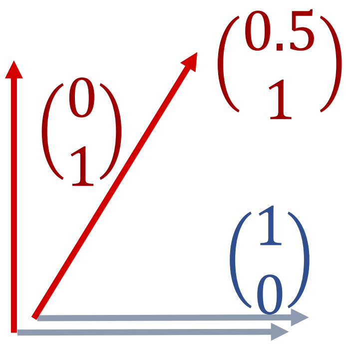
$$\mathbf{M}_{refl} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

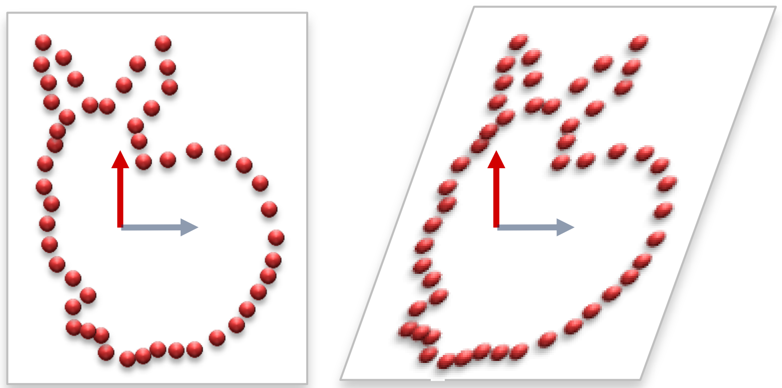
Reflection Axis

General case

$$\mathbf{S}_\lambda : \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{S}_\lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Shearing





$$\mathbf{M}_{shear} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

General Case

You can combine all of these

Example: General axis of rotation

- First rotate rotation axis to x-axis
- Rotate around x
- Rotate back

Question

- How to combine multiple matrix multiplications?

Combining Transformations

Matrix Products

BASIC

basic topics

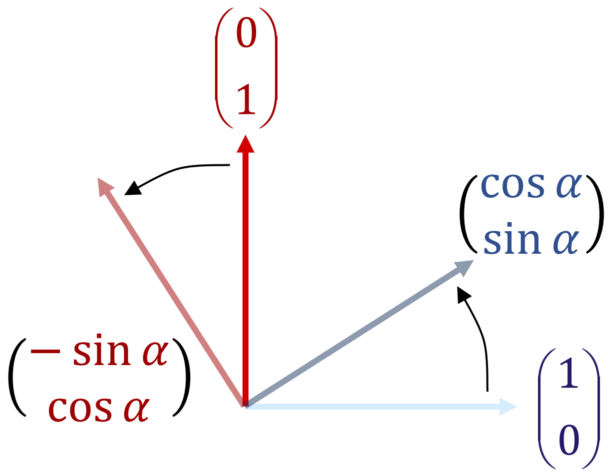
study completely

Matrix Multiplication

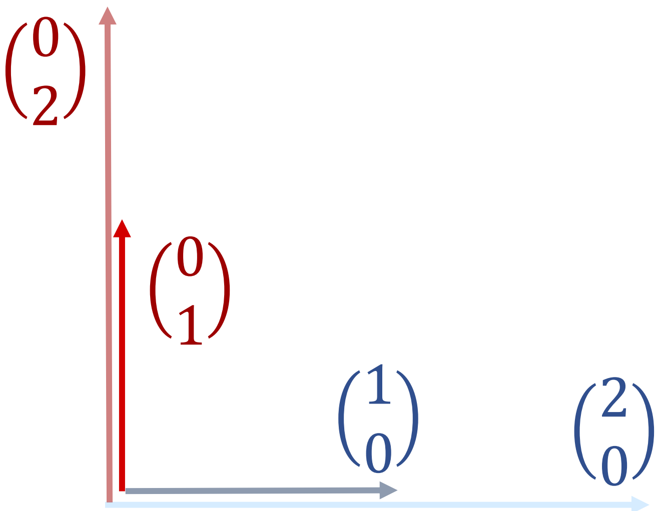
Execute multiple linear maps, one after another

- Written as product
- $(\mathbf{B} \cdot \mathbf{A}) \cdot \mathbf{x}$:
  - Apply $\mathbf{A}$ to $\mathbf{x}$ first
  - Then $\mathbf{B}$
  - $(\mathbf{B} \cdot \mathbf{A})$ is again a matrix

How does it work?



A

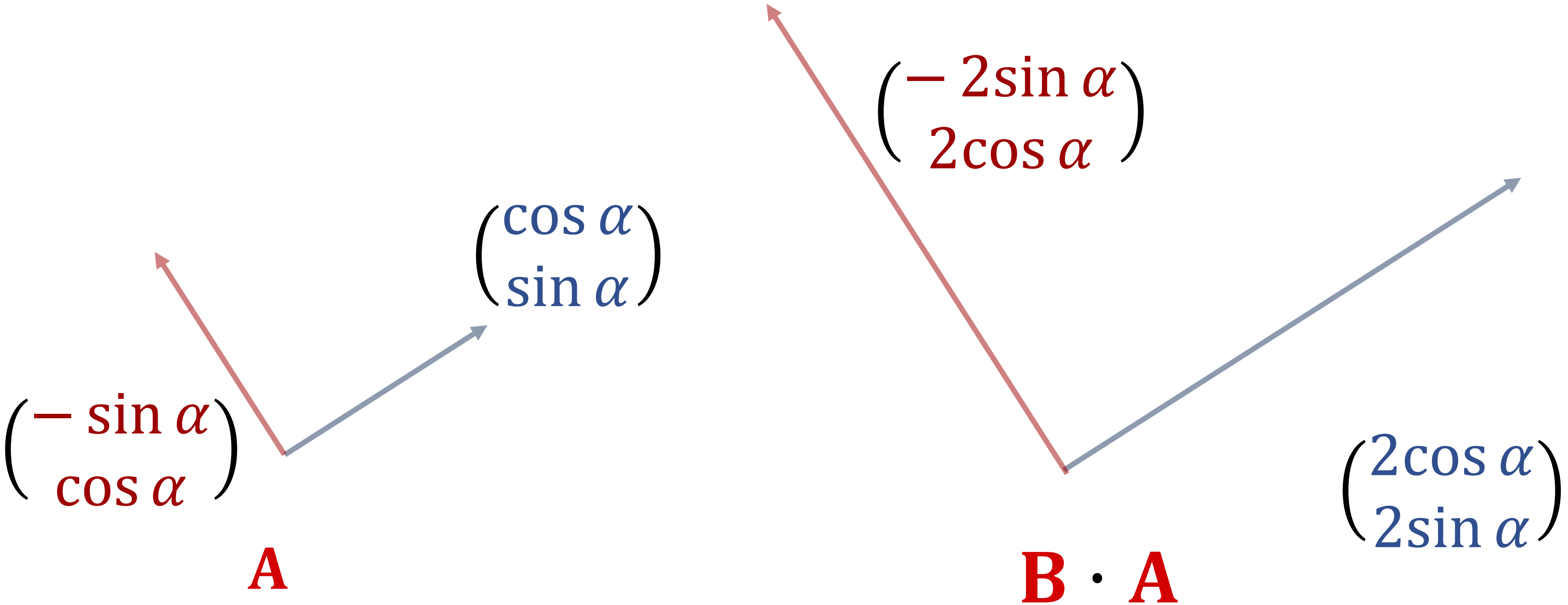


B

Consider $(\mathbf{B} \cdot \mathbf{A})$:

- Rotate first (**A**)
- Then scale (**B**)

How does it work?

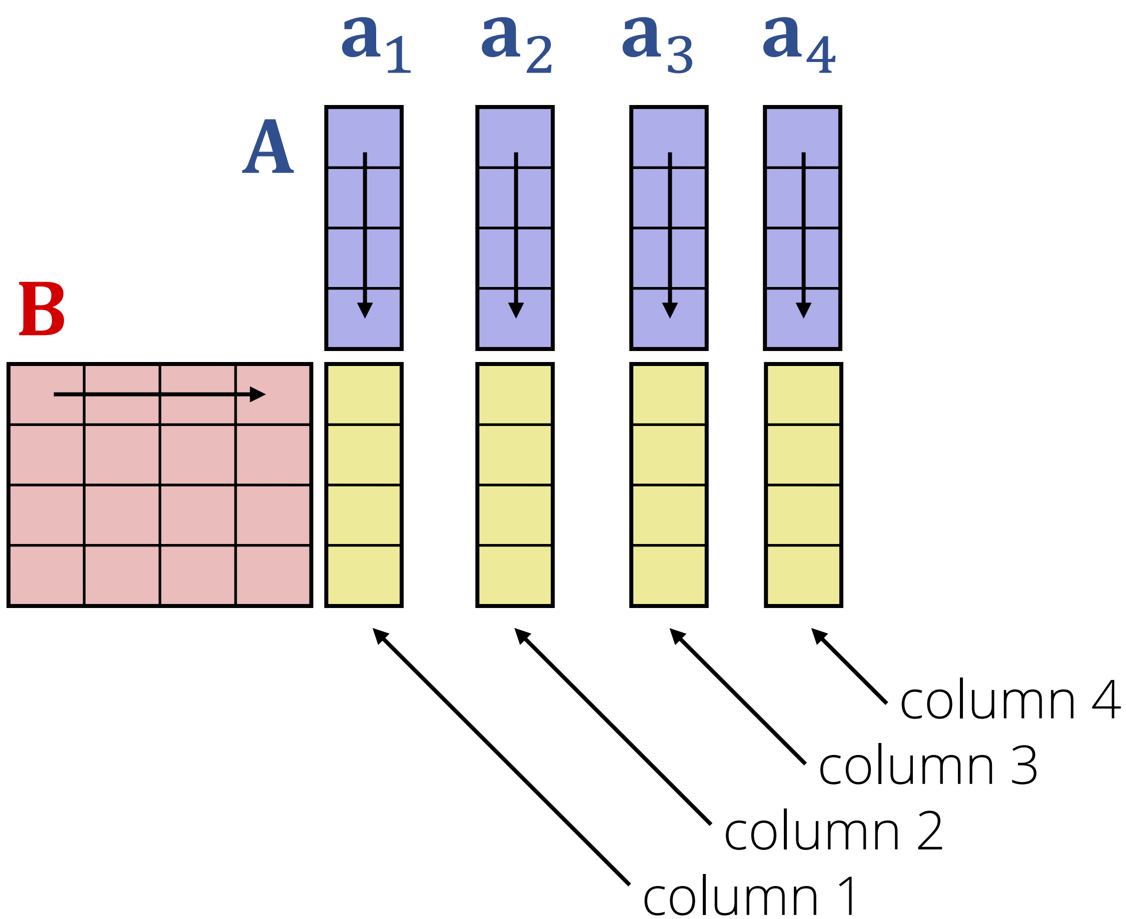


How to compute $(\mathbf{B} \cdot \mathbf{A})$?

- Transform basis vectors
- Transform again

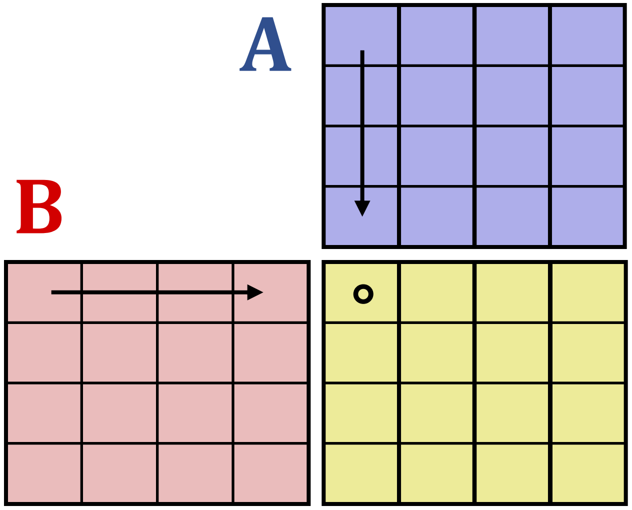
Matrix Multiplication

Matrix product:



Matrix Multiplication

Matrix product:



Matrix Multiplication

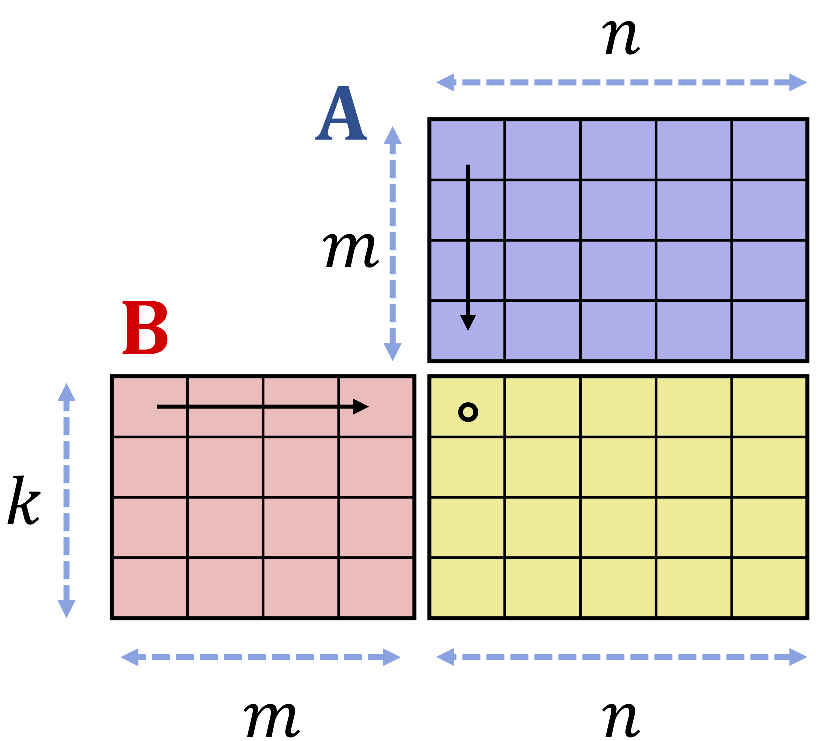
General matrix products:

- **B** · **A**: possible if #Row(**A**) = #Columns(**B**)

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & & \vdots \\ b_{k,1} & \cdots & b_{k,m} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} r_{1,1} & \cdots & r_{1,n} \\ \vdots & & \vdots \\ r_{k,1} & \cdots & r_{k,n} \end{bmatrix}$$



$\mathbf{R} = \mathbf{B} \cdot \mathbf{A}$

$$r_{i,j} = \sum_{q=1}^{m} a_{q,j} \cdot b_{i,q}$$

Rules for Matrix Multiplication

Settings
$\lambda \in \mathbb{R}$
$\mathbf{A}, \mathbf{B}, \mathbf{C}$ - matrices
$\mathbf{v}, \mathbf{w}$ - vectors

Matrix-Multiplication

- Associative

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$$

- Includes vector-multiplication

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{v} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{v})$$

- In general, not commutative:

It might be that $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

- Linear

$$\mathbf{A} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{A} \cdot \mathbf{v} + \mathbf{A} \cdot \mathbf{w}$$
$$\mathbf{A} \cdot (\lambda \cdot \mathbf{v}) = \lambda \cdot (\mathbf{A} \cdot \mathbf{v})$$

(Remark: *linearity is used to define linear maps axiomatically*)

Reversing Transformations

Matrix Inversion

CORE

core topics
important

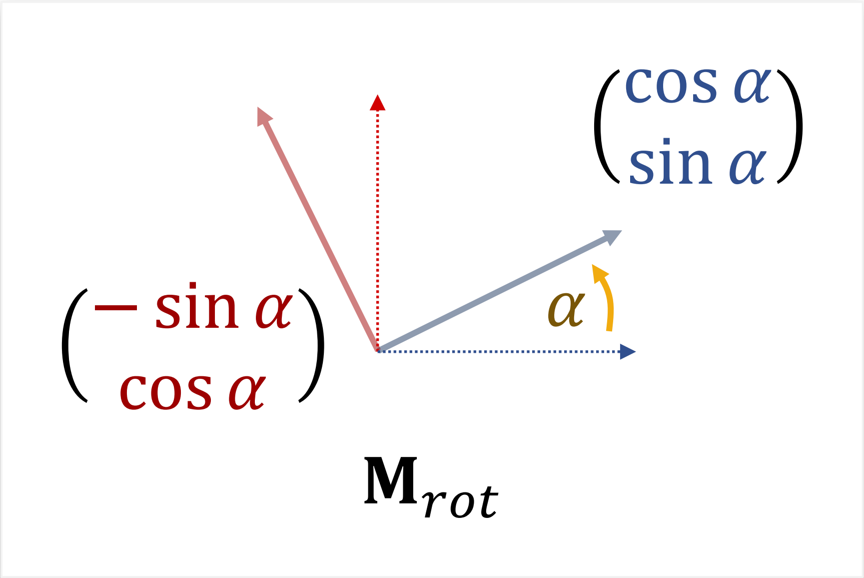
Inverse Matrix

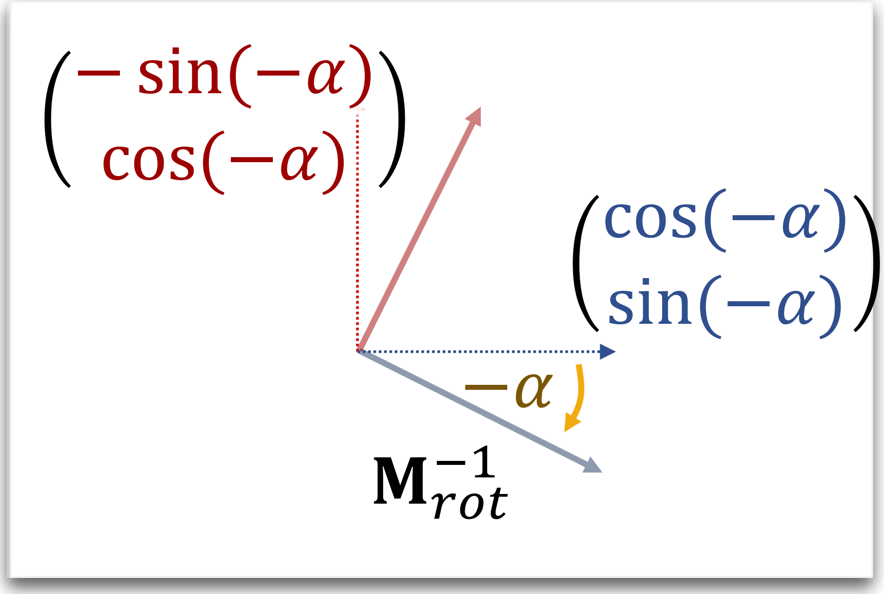
Can we find the inverse matrix?

- “Undo effect”
- Formally

$$\mathbf{M}^{-1} \cdot \mathbf{M} = \mathbf{I}$$

Inverse Matrix





Examples

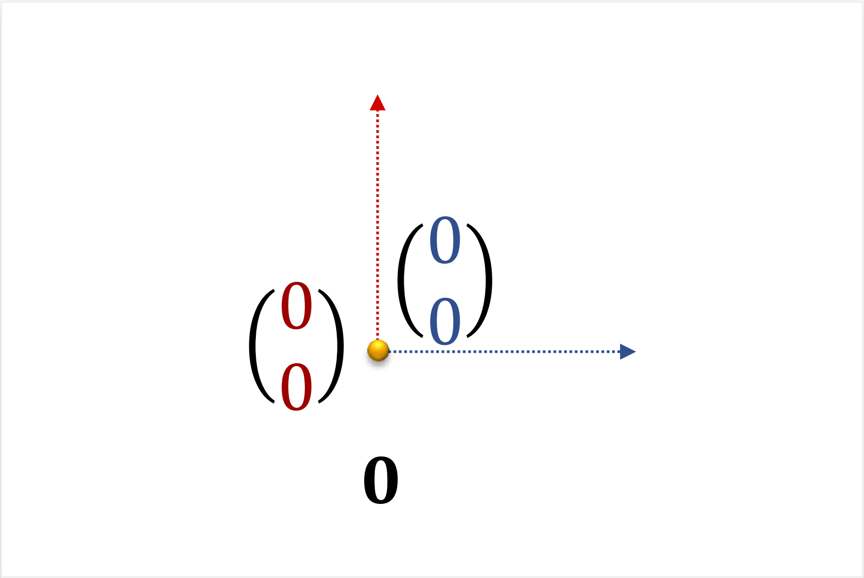
- Rotation matrix

$$\mathbf{M}_{rot} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

- Inverse?

$$\mathbf{M}_{rot}^{-1} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix}$$

Inverse Matrix



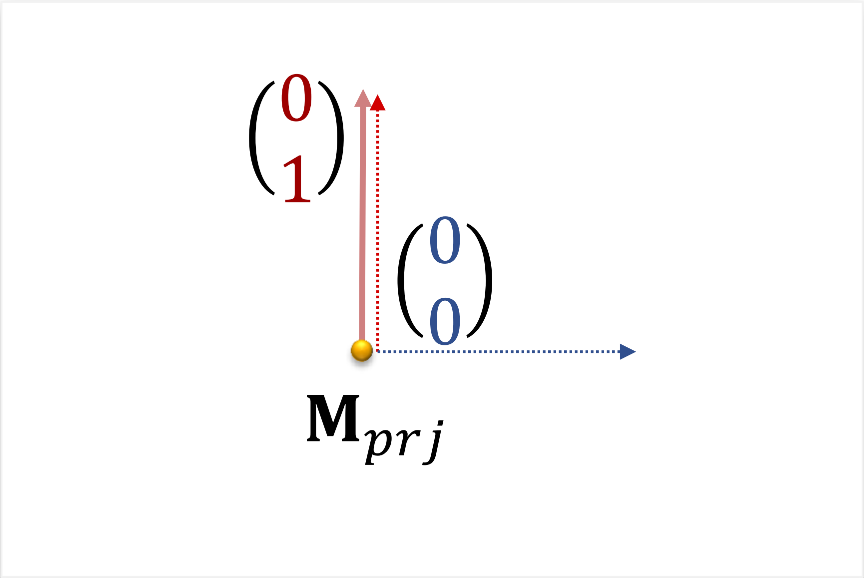
Examples

- Null matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Inverse?

Inverse Matrix



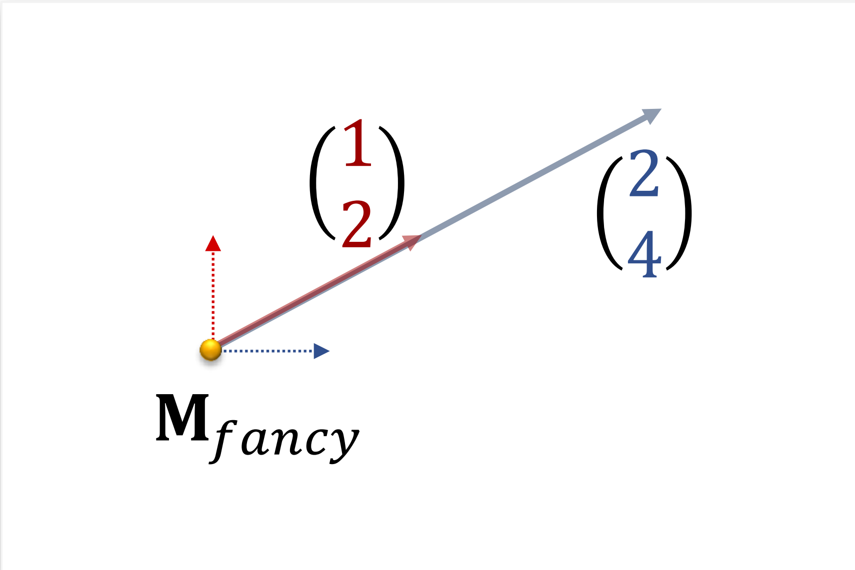
Examples

- Projection matrix (remove x-component)

$$\mathbf{M}_{prj} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- Inverse?

Inverse Matrix



Examples

- Projection matrix (remove x-component)

$$\mathbf{M}_{fancy} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

- Inverse?

Invertible Matrices

Invertible matrices

- Are always square (#rows = #columns)
- In addition
  - Columns are linearly independent

Equivalent characterizations:

- Square and rows are linearly independent
- Columns form basis of vector space
- Rows form basis of vector space

Invertible Matrices

Rank

- Number of linearly independent columns
- Dimension of $\mathrm{span}\{\mathbf{column\_vectors}\}$

Theorem

- Rank = number of linearly independent rows

Full rank

- $\mathrm{rank}(\mathbf{M}) = \dim(V)$
- Then: $\mathbf{M}$ is invertible

Linear Systems of Equations

First consider simpler case

- Say, we know that

$$\mathbf{M} \cdot \mathbf{x} = \mathbf{y}$$

  - Square matrix $\mathbf{M} \in \mathbb{R}^{d \times d}$
  - Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d \times d}$

Knowns & Unknowns

- We are given $\mathbf{M}$, $\mathbf{y}$
- We should compute $\mathbf{x}$
- Linear system of equations

Linear Systems of Equations

Linear System of Equations

$$\mathbf{M} \cdot \mathbf{x} = \mathbf{y}$$

$$\Leftrightarrow$$

$$\begin{bmatrix} m_{1,1} & \cdots & m_{1,d} \\ \vdots & & \vdots \\ m_{d,1} & \cdots & m_{d,d} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix}$$

$$\Leftrightarrow$$

$$m_{1,1}x_1 + \cdots + m_{1,d}\, x_d = y_1$$

$$\text{and } m_{2,1}x_1 + \cdots + m_{2,d}\, x_d = y_2$$

$$\vdots$$

$$\text{and } m_{d,1}x_1 + \cdots + m_{d,d}\, x_d = y_d$$

Gaussian Elimination

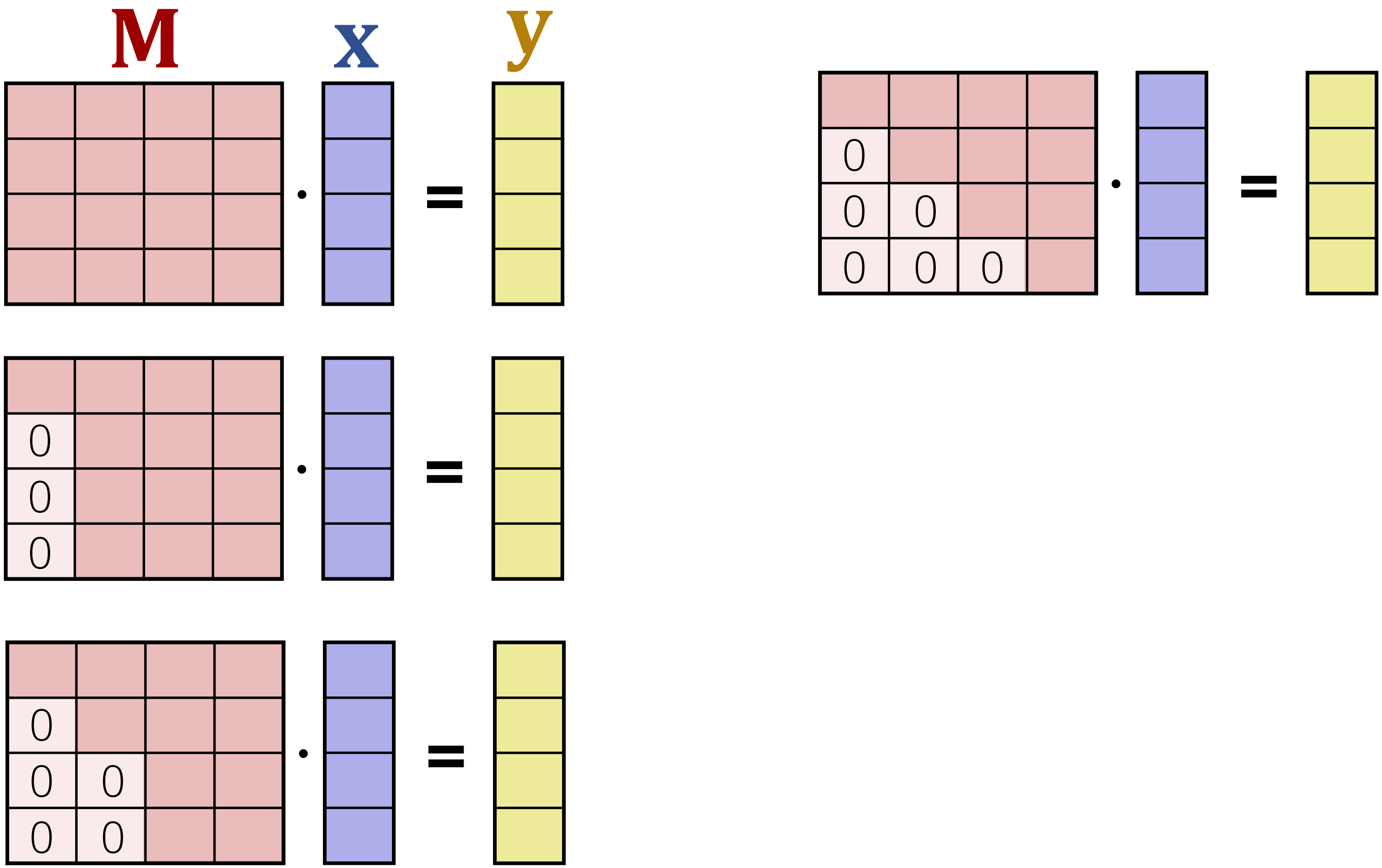
Linear System

$$
\begin{aligned}
& m_{1,1} x_1 + \cdots + m_{1,d}\, x_d = y_1 \\
\land\ & m_{2,1} x_1 + \cdots + m_{2,d}\, x_d = y_2 \\
& \vdots \\
\land\ & m_{d,1} x_1 + \cdots + m_{d,d}\, x_d = y_d
\end{aligned}
$$

Row Operations

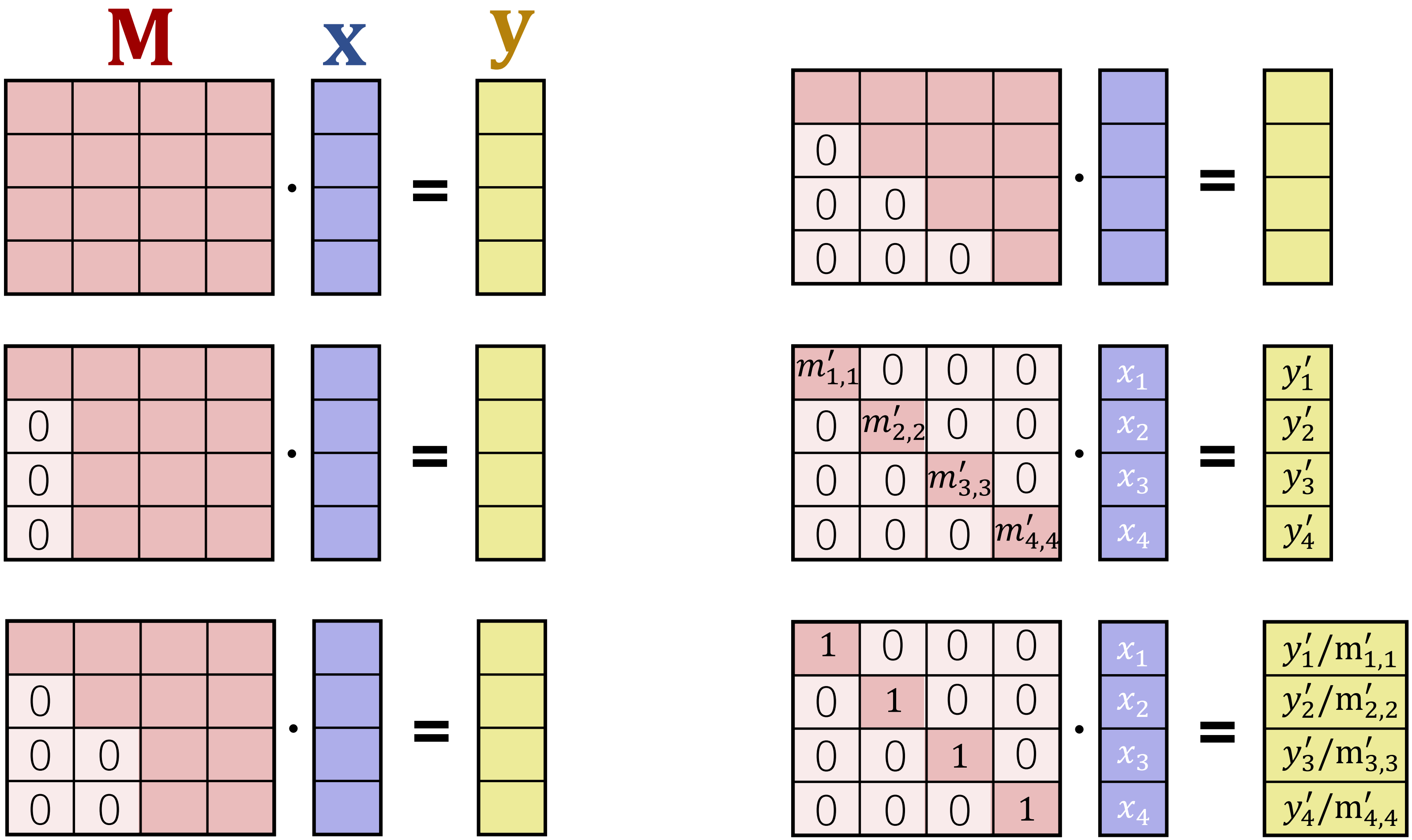
- Swap rows $r_i,\ r_j$
- Scale row r_i by factor $\lambda \neq 0$
- Add multiple of row r_i to row r_j, $i \neq j$
  (i.e., $r_i \mathrel{+}= \lambda r_j$)

Convert to Upper Triangle Matrix



(use row-operations)

Convert to Diagonal Matrix



(use row-operations)

Gauss-Algorithm

Gauss-Algorithm

- Substract rows to cancel front-coefficient
  - Create upper triangle matrix first
  - Then create diagonal matrix
- If current row starts with 0
  - Swap with another row
  - If all rows start with 0: matrix not invertible
- Diagonal form: Solution can be read-off
- Data structure
  - Modify matrix **M**, “right-hand-side” **y**.
  - **x** remains unknown (no change)

Matrix Inverse

Solve for

$$\mathbf{M} \cdot \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{M} \cdot \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \quad \mathbf{M} \cdot \mathbf{x}_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

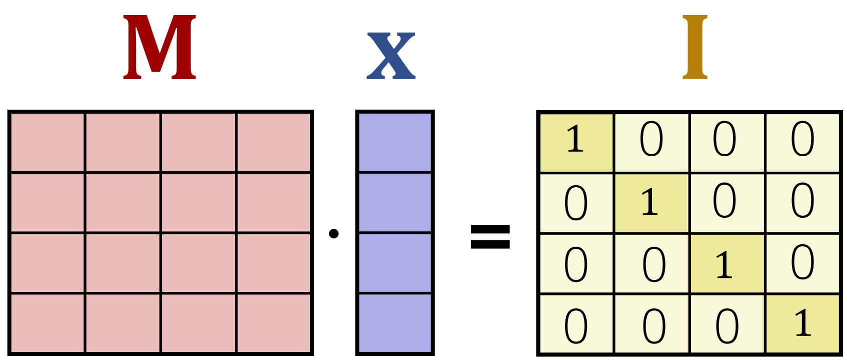
- The resulting $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d$ are the columns of $\mathbf{M}^{-1}$:

$$\mathbf{M}^{-1} = \begin{pmatrix} | & & | \\ \mathbf{x}_1 & \cdots & \mathbf{x}_d \\ | & & | \end{pmatrix}$$

Matrix Inverse

Algorithm

- Simultaneous Gaussian elimination
- Start as follows:



- Handle all right-hand sides simultaneously
- After Gauss-algorithm, the right-hand matrix is the inverse

Alternative: Kramer's Rule

Small Matrices

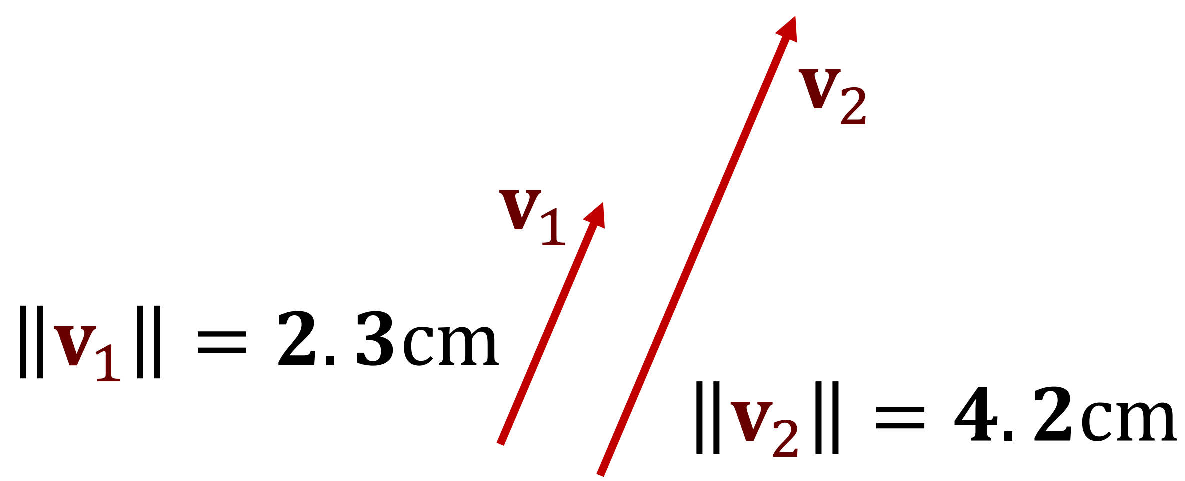
- Direct formula based on determinants
- "Kramer's rule"
- (more later)
  - Naive implementation has run-time $\mathcal{O}(d!)$
    - Gauss: $\mathcal{O}(d^3)$
  - Not advised for $d > 3$

More Vector Operations:
Scalar Products

BASIC

basic topics
study completely

Additional Vector Operations

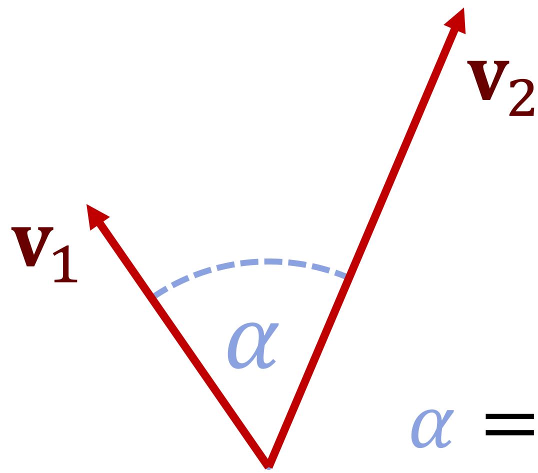


Length of Vectors

"length" or "norm"

$||\mathbf{v}||$ yields real number ≥ 0

Additional Vector Operations

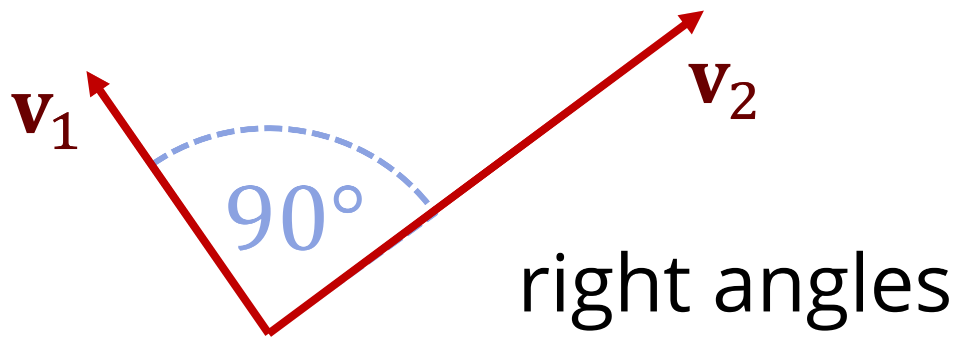


$$\alpha = \angle(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{33°}$$

Angle between Vectors

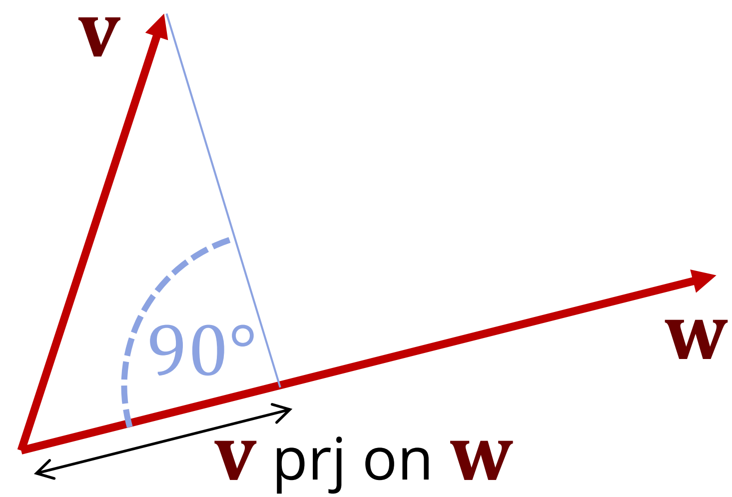
angle $\angle(\mathbf{v}_1, \mathbf{v}_2)$
yields real number
$[0, \ldots, 2\pi) = [0, \ldots, 360°)$

Additional Vector Operations



Angle between Vectors

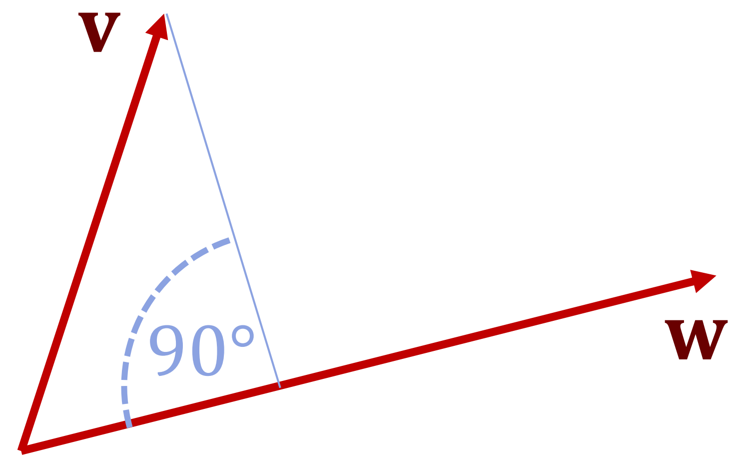
Additional Vector Operations



Projection

Projection: determine
length of **v** along direction of **w**

Additional Vector Operations



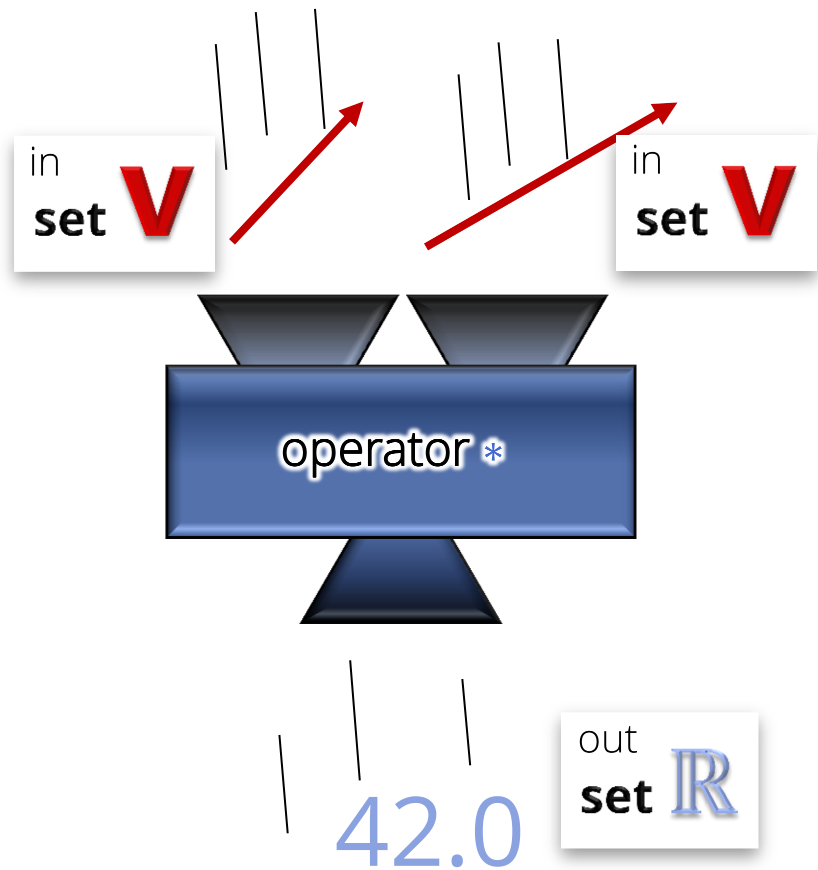
Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

also: $\langle \mathbf{v}, \mathbf{w} \rangle$

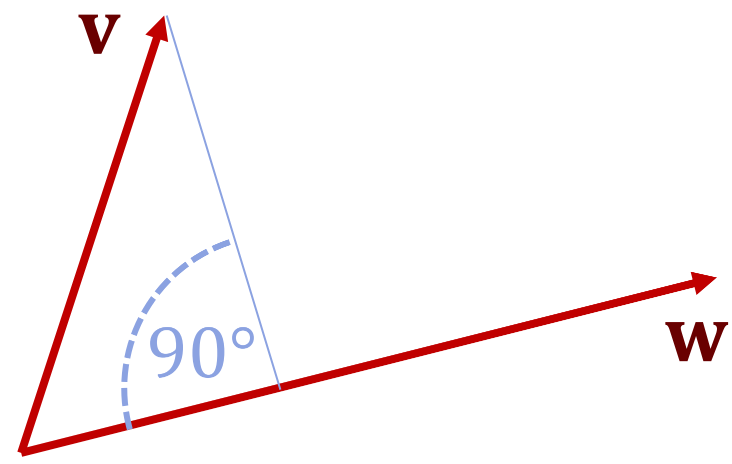
*) also known as *inner product* or *dot-product*

Signature



Scalar Product
(dot product, inner-product)

Additional Vector Operations



Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

also: $\langle \mathbf{v}, \mathbf{w} \rangle$

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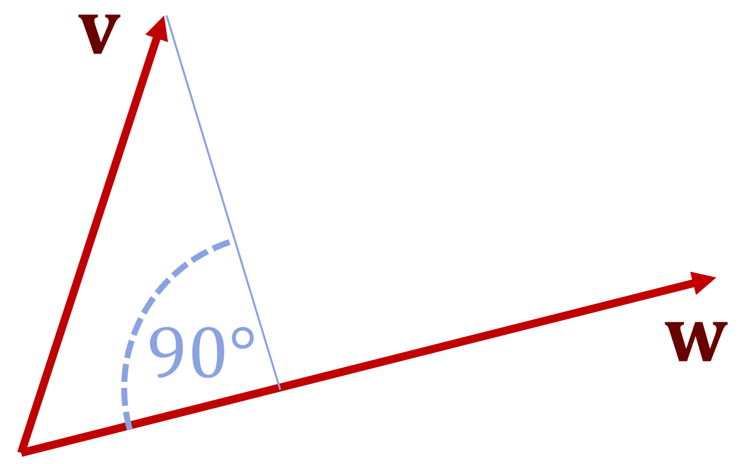
Additional Vector Operations



Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

Comprises: length, projection, angles

*) also known as *inner product* or *dot-product*

Additional Vector Operations

Length: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Angle: $\angle(\mathbf{v}, \mathbf{w}) = \arccos(\mathbf{v} \cdot \mathbf{w})$

Projection: „$\mathbf{v}$ prj on $\mathbf{w}$" $= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|}$

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

Comprises: length, projection, angles

Algebraic Representation

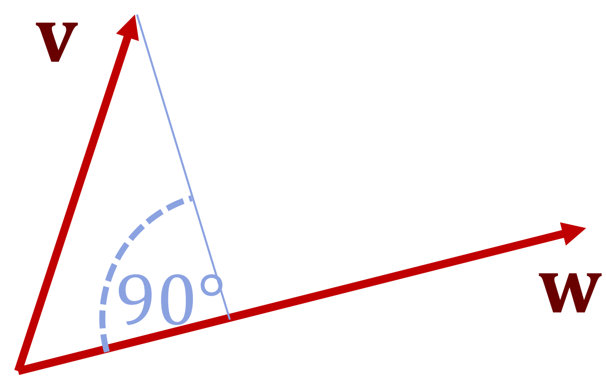
(Implementation)

BASIC

basic topics

study completely

Scalar Product



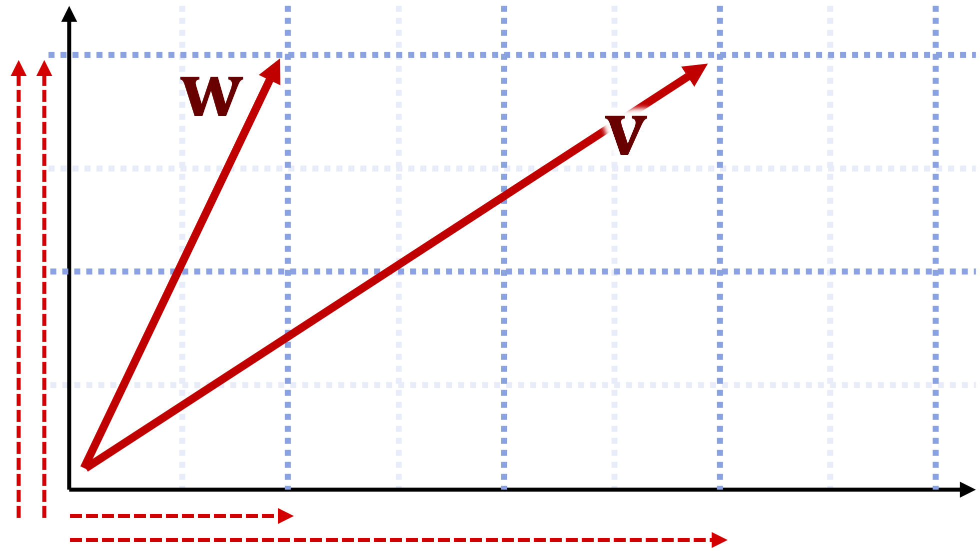
Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := v_1 \cdot w_1 + v_2 \cdot w_2$$

Theorem:

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

Scalar Product



$$\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Scalar product

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Scalar Product

2D Scalar product

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := v_1 \cdot w_1 + v_2 \cdot w_2$$

d-dim scalar product

$$\mathbf{v} \cdot \mathbf{w} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} := v_1 \cdot w_1 + \cdots + v_d \cdot w_d$$

Algebraic Properties

Settings

$\lambda \in \mathbb{R}$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^d$

Properties

- Symmetry (commutativity)

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

- Bilinearity

$$\langle \lambda \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \lambda \mathbf{w} \rangle$$

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

(symmetry: same for second argument)

- Positive definite

$$\langle \mathbf{u}, \mathbf{u} \rangle \geq 0, \qquad [\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{0}] \Rightarrow [\mathbf{u} = \mathbf{0}]$$

These three: axiomatic definition

Attention!

Do not mix

- Scalar-vector product
- Inner (scalar) product

In general

$$\langle \mathbf{x}, \mathbf{y} \rangle \cdot \mathbf{z} \neq \mathbf{x} \cdot \langle \mathbf{y}, \mathbf{z} \rangle$$

Beware of notation:

$$(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} \neq \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$$

(no violation of associativity: different operations; details later)

Applications of the Scalar Product

CORE

core topics
important

Applications

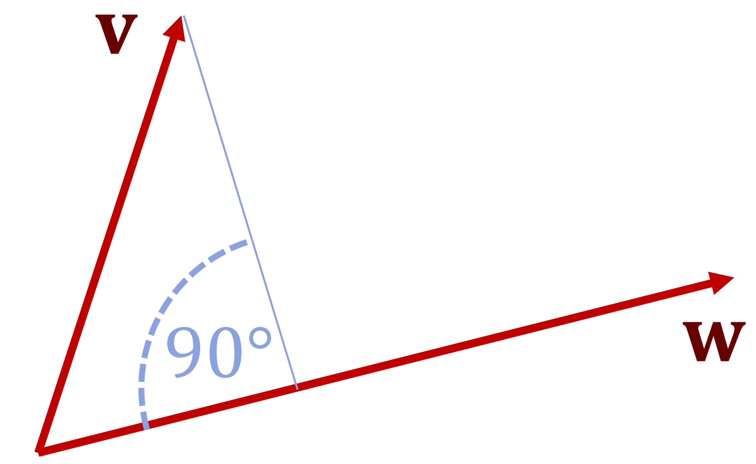
Obvious applications

- Measuring length
- Measuring angles
- Projections

More complex applications

- Creating orthogonal (90°) pairs of vectors
- Creating orthogonal bases

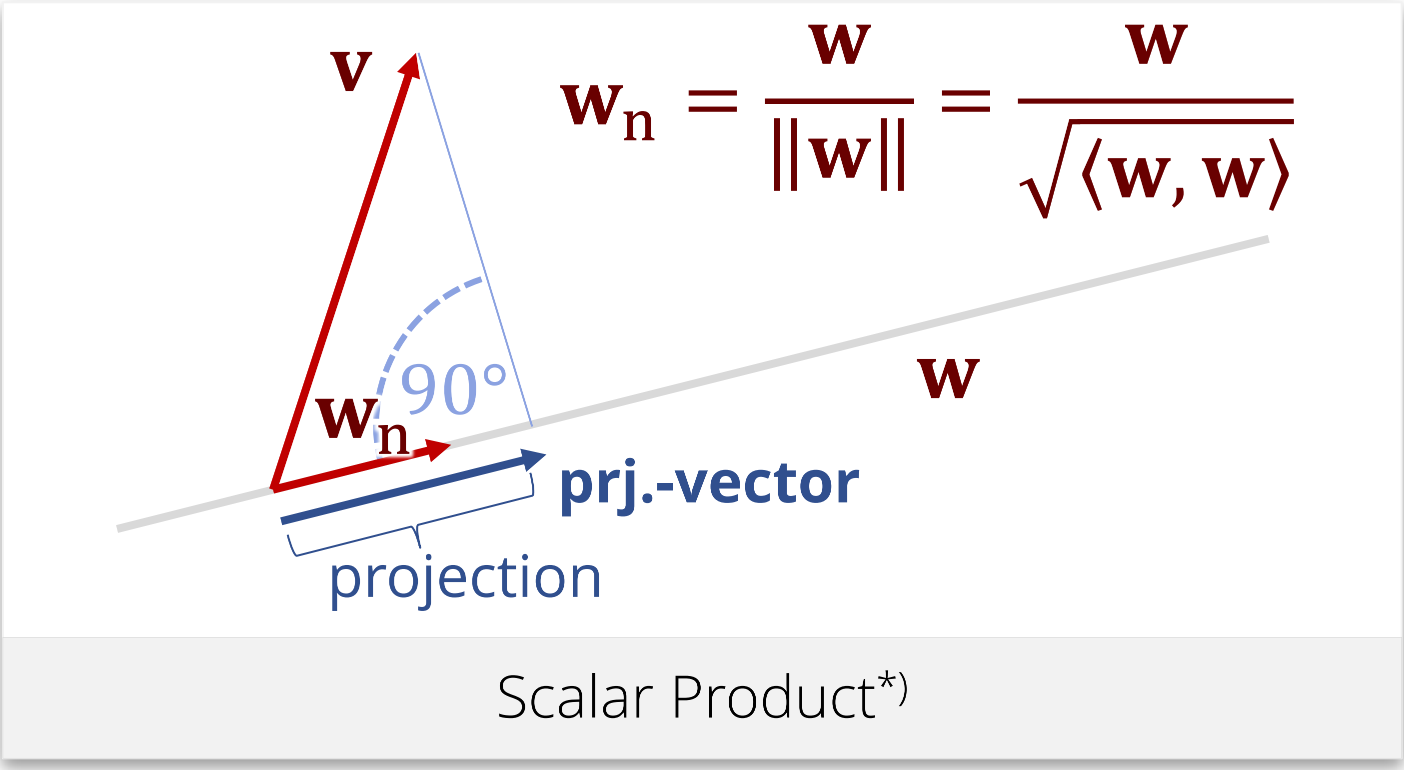
Projection



Scalar Product*)

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cdot \cos \angle(\mathbf{v}, \mathbf{w})$$

Projection

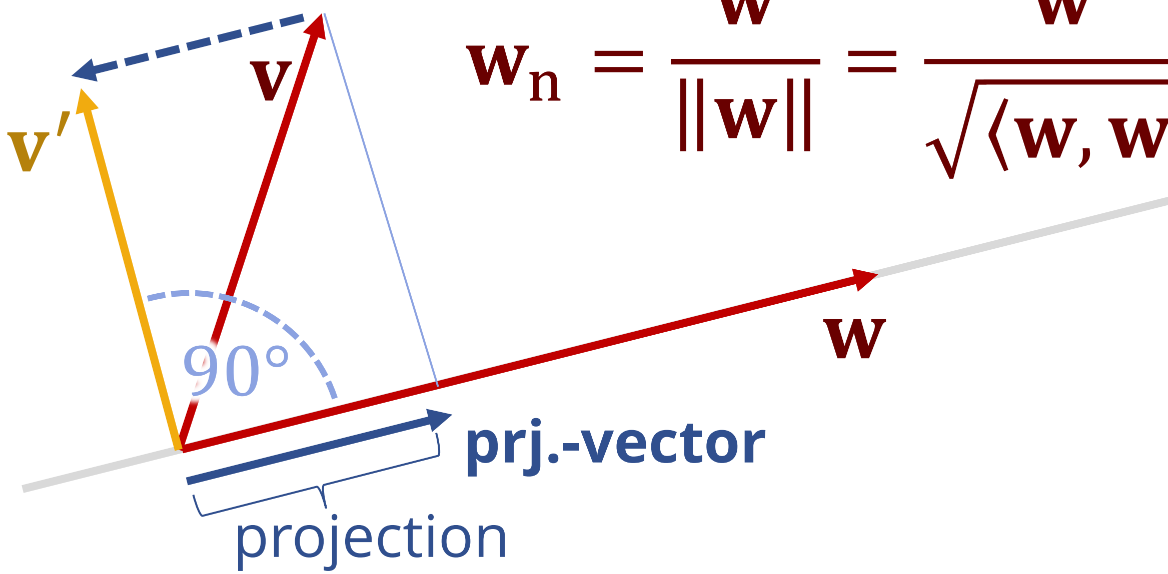


Projection: $\mathbf{v} \cdot \frac{\mathbf{w}}{\sqrt{\mathbf{w} \cdot \mathbf{w}}}$

Prj.-Vector: $\left\langle \mathbf{v}, \frac{\mathbf{w}}{\sqrt{\langle \mathbf{w},\mathbf{w} \rangle}} \right\rangle \cdot \frac{\mathbf{w}}{\sqrt{\langle \mathbf{w},\mathbf{w} \rangle}}$

$= \langle \mathbf{v}, \mathbf{w} \rangle \cdot \frac{\mathbf{w}}{\langle \mathbf{w},\mathbf{w} \rangle}$

Orthogonalization



$$\mathbf{w}_n = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{\mathbf{w}}{\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}}$$

Scalar Product*)

Orthogonalize $\mathbf{v}$ wrt. $\mathbf{w}$:

$$\mathbf{v}' = \mathbf{v} - \langle \mathbf{v}, \mathbf{w} \rangle \cdot \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle}$$

Orthogonalization

Scalar Product*)

Orthogonalize $\mathbf{v}$ wrt. $\mathbf{w}$:

$$\mathbf{v}' = \mathbf{v} - \langle \mathbf{v}, \mathbf{w} \rangle \cdot \frac{\mathbf{w}}{\langle \mathbf{w}, \mathbf{w} \rangle}$$

Gram-Schmidt Orthogonalization

Orthogonal basis

- All vectors in 90° angle to each other

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0 \text{ for } i \neq j$$

Create orthogonal bases

- Start with arbitrary one
- Orthogonalize $\mathbf{b}_2$ by $\mathbf{b}_1$
- Orthogonalize $\mathbf{b}_3$ by $\mathbf{b}_1$, then by $\mathbf{b}_2$
- Orthogonalize $\mathbf{b}_4$ by $\mathbf{b}_1$, then by $\mathbf{b}_2$, then by $\mathbf{b}_3$
- …

Orthonormal Basis

Orthonormal bases

- Orthogonal and all vectors have unit length

Computation

- Orthogonalize first
- Then scale each vector $\mathbf{b}_i$ by $1/\|\mathbf{b}_i\|$.

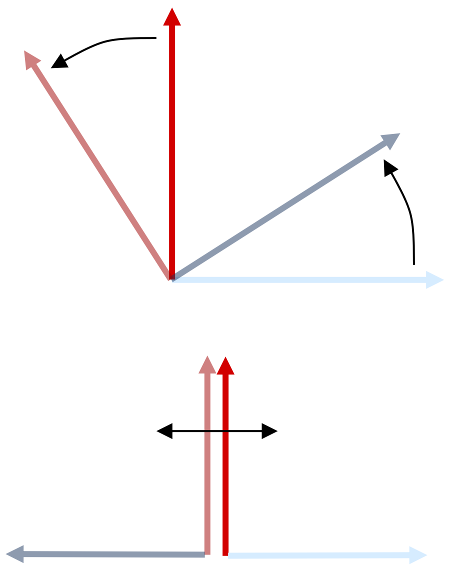
Matrices

Orthogonal Matrices

- A matrix with *orthonormal* columns is called *orthogonal* matrix
  - Yes, this terminology is not quite logical...

Orthogonal Matrices are always

- Rotation matrices
- Or reflection matrices
- Or products of the two



Further Operations

CORE

core topics

important

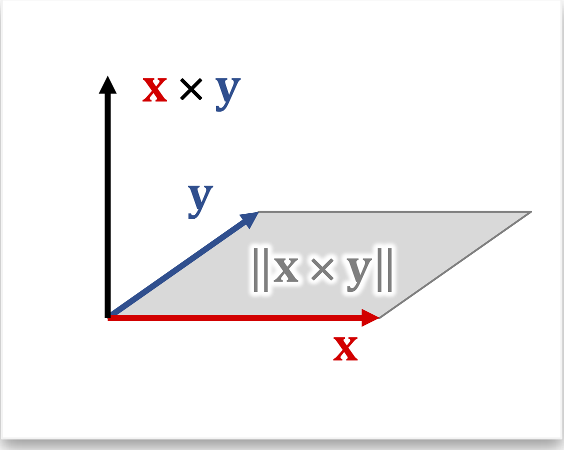
Cross Product

Cross-Product: Exists Only For 3D Vectors!

- $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$
- $\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$

Geometrically: Theorem

- $\mathbf{x} \times \mathbf{y}$ orthogonal to $\mathbf{x}, \mathbf{y}$
- Right-handed system $(\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y})$
- $\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \sin\angle(\mathbf{x}, \mathbf{y})$



Cross-Product Properties

Bilinearity

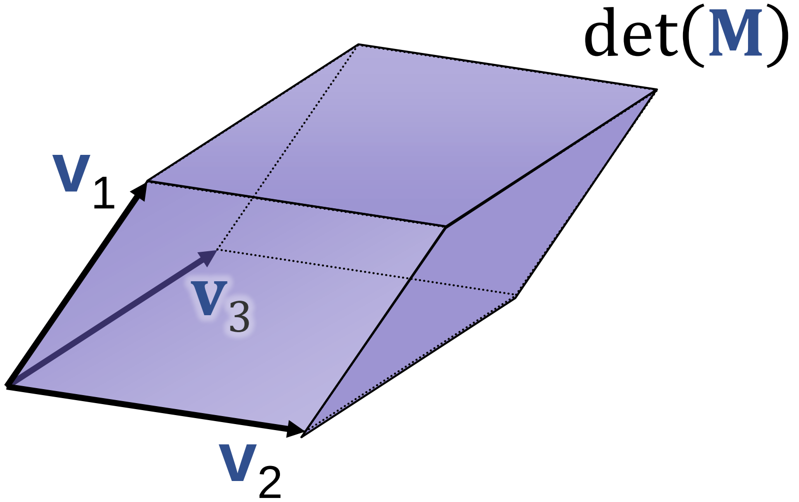
- Distributive: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- Scalar-Mult.: $(\lambda \mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\lambda \mathbf{v}) = \lambda (\mathbf{u} \times \mathbf{v})$

But beware of

- **Anti**-Commutative: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- **Not** associative;
  we can have $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

Determinants

$$\mathbf{M} = \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix}$$



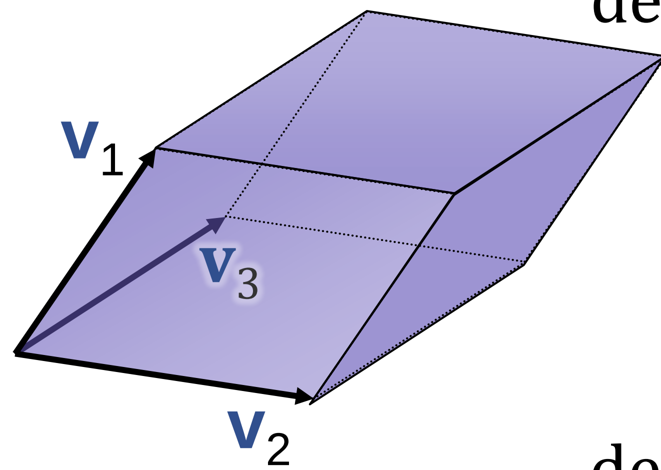
Determinants

- Square matrix **M**
- det(**M**) = |**M**| = volume of *parallelepiped* of column vectors

Determinants

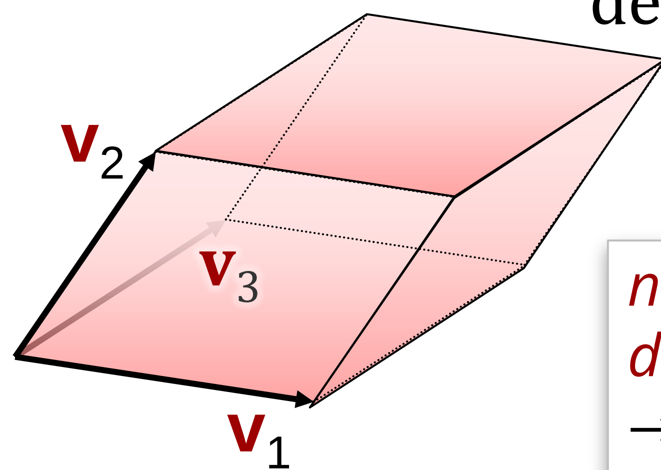
$\det(\mathbf{M}) > 0$

$$\mathbf{M} = \begin{pmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{pmatrix}$$



$\det(\mathbf{M}') < 0$

$$\mathbf{M}' = \begin{pmatrix} | & | & | \\ \mathbf{v}_2 & \mathbf{v}_1 & \mathbf{v}_3 \\ | & | & | \end{pmatrix}$$



negative determinant
→ map contains reflection

Sign:

- Positive for right handed coordinates
- Negative for left-handed coordinates

Properties

A few properties:

- $\det(\mathbf{A})\det(\mathbf{B}) = \det(\mathbf{A}\cdot\mathbf{B})$
- $\det(\lambda\mathbf{A}) = \lambda^d \det(\mathbf{A})$ ($d \times d$ matrix $\mathbf{A}$)
- $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$
- $\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})$
- $[\det(\mathbf{A}) \neq 0] \Leftrightarrow [\mathbf{A} \text{ invertible}]$
- Efficient computation using Gaussian elimination

sign flips!
→ reflections cancel each other (parity)

Computing Determinants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Recursive Formula

- Sum over first row
- Multiply element there with subdeterminant
  - Subdeterminant : Leave out row and column of selected element
  - Recursion ends with $|a| = a$
- Alternate signs $+/-/+/-/\ldots$

signs

$$\begin{vmatrix} +a & -b & +c \\ d & e & f \\ g & h & i \end{vmatrix}$$

subdeterminants

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \longrightarrow \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

$$|a| = a$$

Beware of $\mathcal{O}(dim!)$ complexity

Computing Determinants

Result in 3D Case

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - ceg - bdi - afh$$

Solving Linear Systems

Consider

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

- Invertible matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$
- Known vector $\mathbf{b} \in \mathbb{R}^{d}$
- Unknown vector $\mathbf{x} \in \mathbb{R}^{d}$

Solution with Determinants (Cramar's rule):

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

$$\mathbf{A}_i = \begin{pmatrix} | & & | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{b} & \cdots & \mathbf{v}_3 \\ | & & | & & | \end{pmatrix}$$

column i

Addendum

Matrix Algebra

ADV

advanced topics

main ideas

Matrix Algebra

Define three operations

- Matrix addition

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{m,1} & \cdots & b_{m,n} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$$

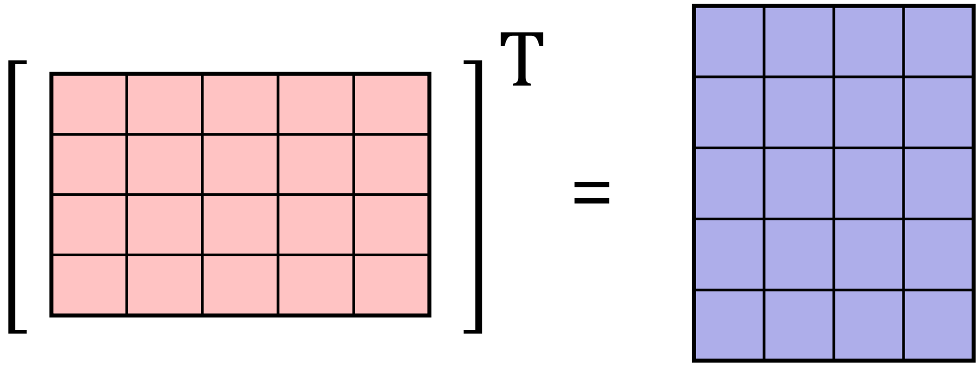
- Scalar matrix multiplication

$$\lambda \cdot \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda \cdot a_{1,1} & \cdots & \lambda \cdot a_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda \cdot a_{m,1} & \cdots & \lambda \cdot a_{m,n} \end{bmatrix}$$

- Matrix-matrix multiplication

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,m} \end{bmatrix} = \begin{bmatrix} \ddots & & ⋰ \\ & \sum_{q=1}^{k} a_{q,j} \cdot b_{i,q} & \\ ⋰ & & \ddots \end{bmatrix}$$

Transposition



Matrix Transposition

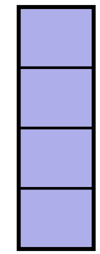
- Swap rows and columns
- Formally:

$$\begin{bmatrix} \ddots & \vdots & \cdot^{\cdot^{\cdot}} \\ \cdots & a_{i,j} & \cdots \\ \cdot^{\cdot^{\cdot}} & \vdots & \ddots \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \ddots & \cdots & \cdot^{\cdot^{\cdot}} \\ \vdots & a_{j,i} & \vdots \\ \cdot^{\cdot^{\cdot}} & \cdots & \ddots \end{bmatrix}$$

Vectors

Vectors

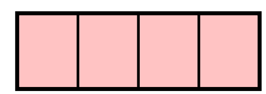
- Column matrices
- Matrix-Vector product consistent



$\mathbf{x} \in \mathbb{R}^d$

Co-Vectors

- "projectors", "dual vectors", "linear forms", "row vectors"
- Vectors to be projected on

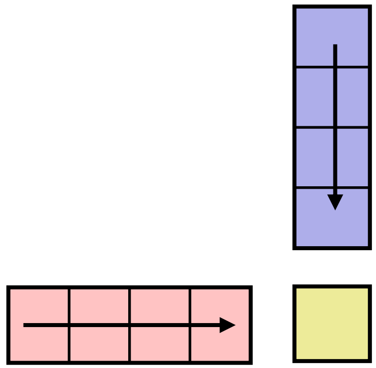


$\mathbf{y}^{\mathrm{T}} \in \mathbb{R}^d$

Transposition

- Convert vectors into projectors and vice versa

Vectors



$$\mathbf{x}^{\mathrm{T}} \cdot \mathbf{y} \to \mathbb{R}$$

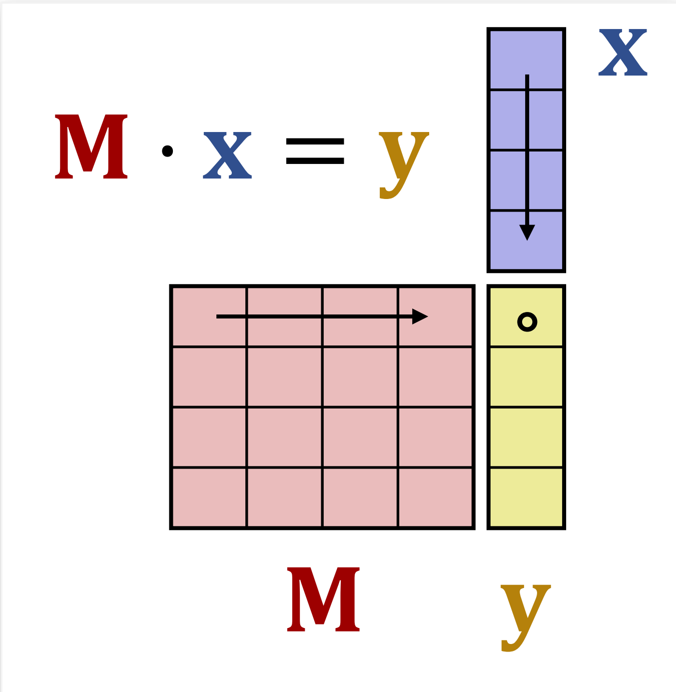
Inner product (as a generalized “projection”)

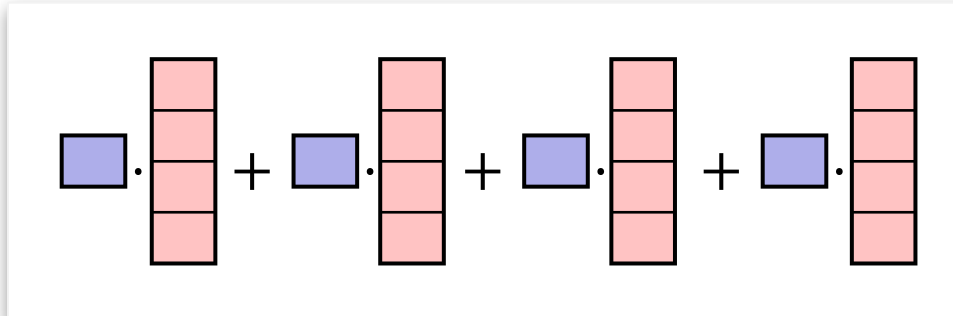
- Matrix-product **column** · **row**

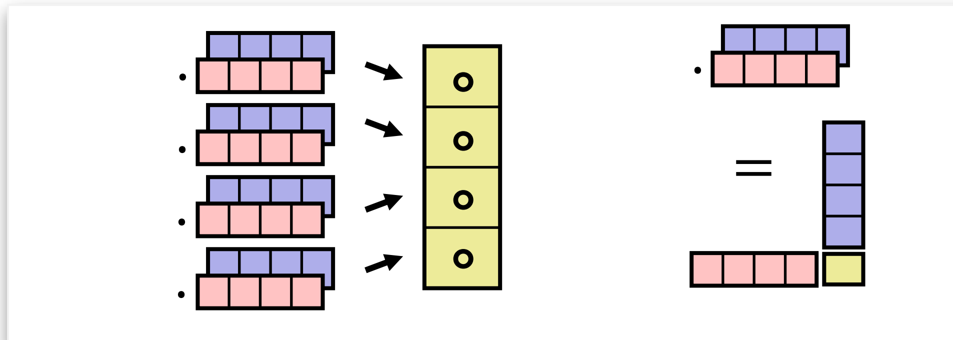
$$„\mathbf{x} \cdot \mathbf{y}“ = \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}$$

- People use all three notations
  - Meaning of “ · ” clear from context

Matrix-Vector Products







Two Interpretations

- Linear combination of column vectors
- Projection on row (co-)vectors

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main ideas

Matrix Algebra

We can add and scalar multiply

- Matrices and vectors (special case)

We can matrix-multiply

- Matrices with other matrices (execute one-after-another)
- Vectors in certain cases (next)

We can “divide” by some (not all) matrices

- Determine inverse matrix
- Full-rank, square matrices only

Algebraic Rules: Addition

Settings

$\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$
(matrices, same size)

Addition: like real numbers ("commutative group")

- Prerequisites:
  - Number of rows match
  - Number of columns match
- Associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Subtraction: $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
- Neutral Op.: $\mathbf{A} + \mathbf{0} = \mathbf{A}$

Algebraic Rules: Scalar Multiplication

Scalar Multiplication: Vector space

- Prerequisites:
  - Always possible
- Repeated Scaling: $\lambda(\mu\mathbf{A}) = \lambda\mu(\mathbf{A})$
- Neutral Operation: $1 \cdot \mathbf{A} = \mathbf{A}$
- Distributivity 1: $\lambda(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$
- Distributivity 2: $(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$

Settings
$\lambda \in \mathbb{R}$
$\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times m}$
(same size)

So far:

- Matrices form vector space
- Just different notation, same semantics!

Algebraic Rules: Multiplication

Multiplication: Non-Commutative Ring / Group

- Prerequisites:
  - Number of columns right = number of rows left
- Associative: $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$
- Not commutative: often $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$
- Neutral Op.: $\mathbf{A} \cdot \mathbf{I} = \mathbf{A}$
- Inverse: $\mathbf{A} \cdot (\mathbf{A}^{-1}) = \mathbf{I}$
  - Additional prerequisite:
    - Matrix must be square!
    - Matrix must have full rank

Set of invertible matrices:

$$GL(d) \subset \mathbb{R}^{d \times d}$$

"general linear group"

Algebraic Rules: Multiplication

Settings

$\mathbf{A} \in \mathbb{R}^{n \times m}$

$\mathbf{B} \in \mathbb{R}^{m \times k}$

$\mathbf{C} \in \mathbb{R}^{k \times l}$

Multiplication: Non-Commutative Ri

- Prerequisites:
  - Number of columns right = number of rows left
- Associative: $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$
- Not commutative: often $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$
- Neutral Op.: $\mathbf{A} \cdot \mathbf{I} = \mathbf{A}$
- Inverse: $\mathbf{A} \cdot (\mathbf{A}^{-1}) = \mathbf{I}$
  - Additional prerequisite:
    - Matrix must be square!
    - Matrix must have full rank

Set of invertible matrices:

$GL(d) \subset \mathbb{R}^{d \times d}$

"general linear group"

Transposition Rules

Transposition

- Addition: $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} + \mathbf{A}^{\mathrm{T}}$
- Scalar-mult.: $(\lambda\mathbf{A})^{\mathrm{T}} = \lambda\mathbf{A}^{\mathrm{T}}$
- Multiplication: $(\mathbf{A}\cdot\mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\cdot\mathbf{A}^{\mathrm{T}}$
- Self-inverse: $\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}$
- (Inversion:) $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$
- Inverse-transp.: $\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$
- Othogonality: $[\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1}] \Leftrightarrow [\mathbf{A}\ \textbf{is orthogonal}]$

Matrix Multiplication

$$\mathbf{A} \cdot \mathbf{B}$$

$$= \begin{pmatrix} - & \mathbf{a}_1 & - \\ & \vdots & \\ - & \mathbf{a}_d & - \end{pmatrix} \cdot \begin{pmatrix} | & & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_d \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} \ddots & & \iddots \\ & \langle \mathbf{a}_i, \mathbf{b}_j \rangle & \\ \iddots & & \ddots \end{pmatrix}$$

- Scalar products of rows and columns

Orthogonal Matrices

Othogonal Matrices

- (i.e., column vectors ortho*normal*)

$$\mathbf{M}^T = \mathbf{M}^{-1}$$

- Proof: previous slide.

Scalar Product

Matrix Algebra:

- Scalar product is a special case

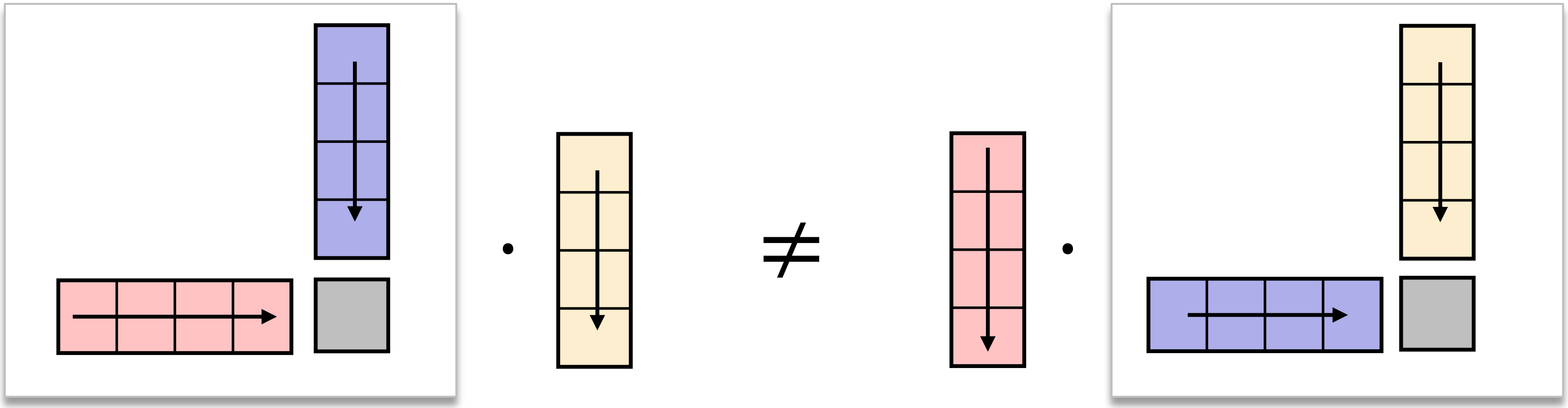
$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}$$

- Caution when mixing with scalar-vector product!

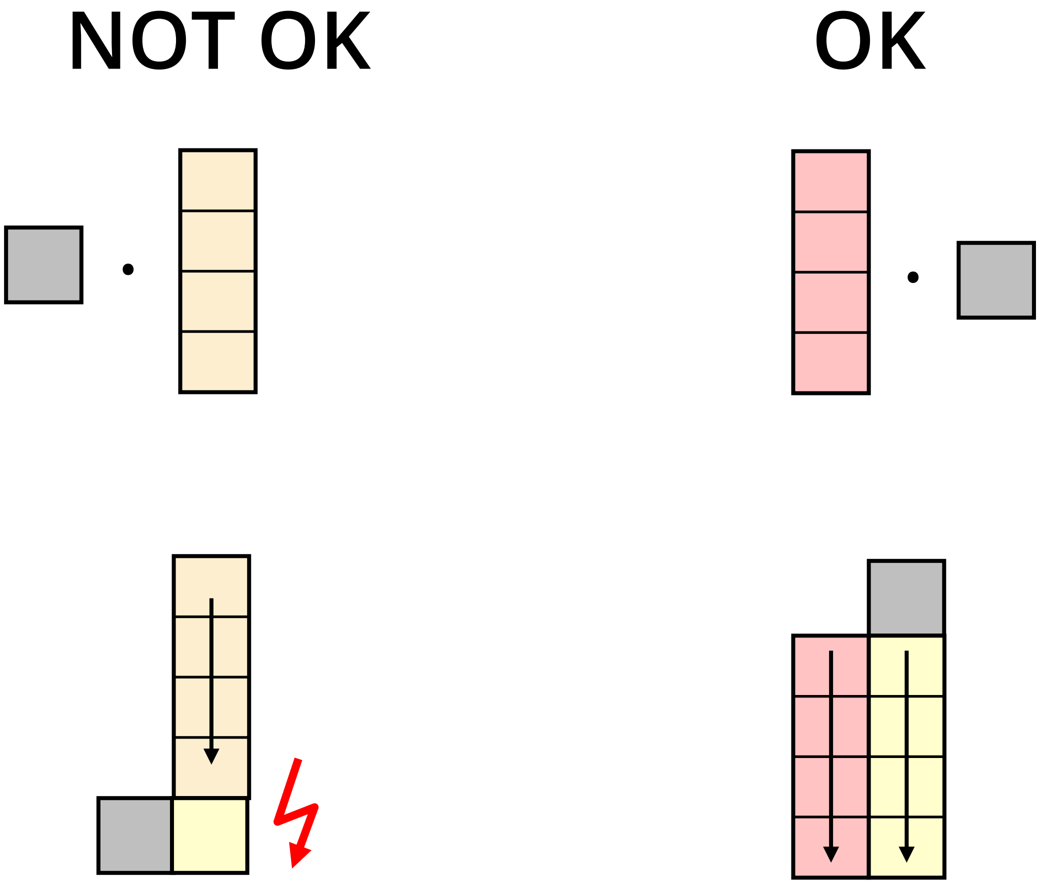
$$\langle \mathbf{x}, \mathbf{y} \rangle \cdot \mathbf{z} \neq \mathbf{x} \cdot \langle \mathbf{y}, \mathbf{z} \rangle$$

$$\left(\mathbf{x}^{\mathrm{T}} \cdot \mathbf{y}\right) \cdot \mathbf{z} \neq \mathbf{x} \cdot \left(\mathbf{y}^{\mathrm{T}} \cdot \mathbf{z}\right)$$

Scalar multiplication not a matrix-product!



Scalar Product

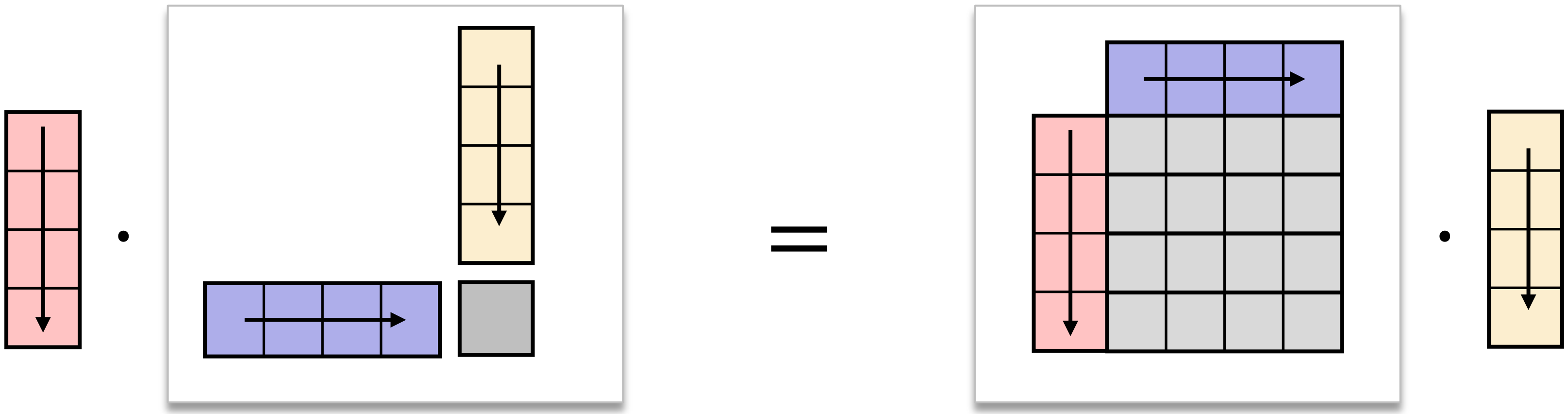


Scalar Product

What does work:

- Associativity with outer product

$$\mathbf{x} \cdot \langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{x} \cdot \left(\mathbf{y}^{\mathrm{T}} \cdot \mathbf{z} \right)$$
$$= \left(\mathbf{x} \cdot \mathbf{y}^{\mathrm{T}} \right) \cdot \mathbf{z}$$



Addendum

Axiomatic Mathematics

(This is not a core topic of the course;
material is provided just for your information.)

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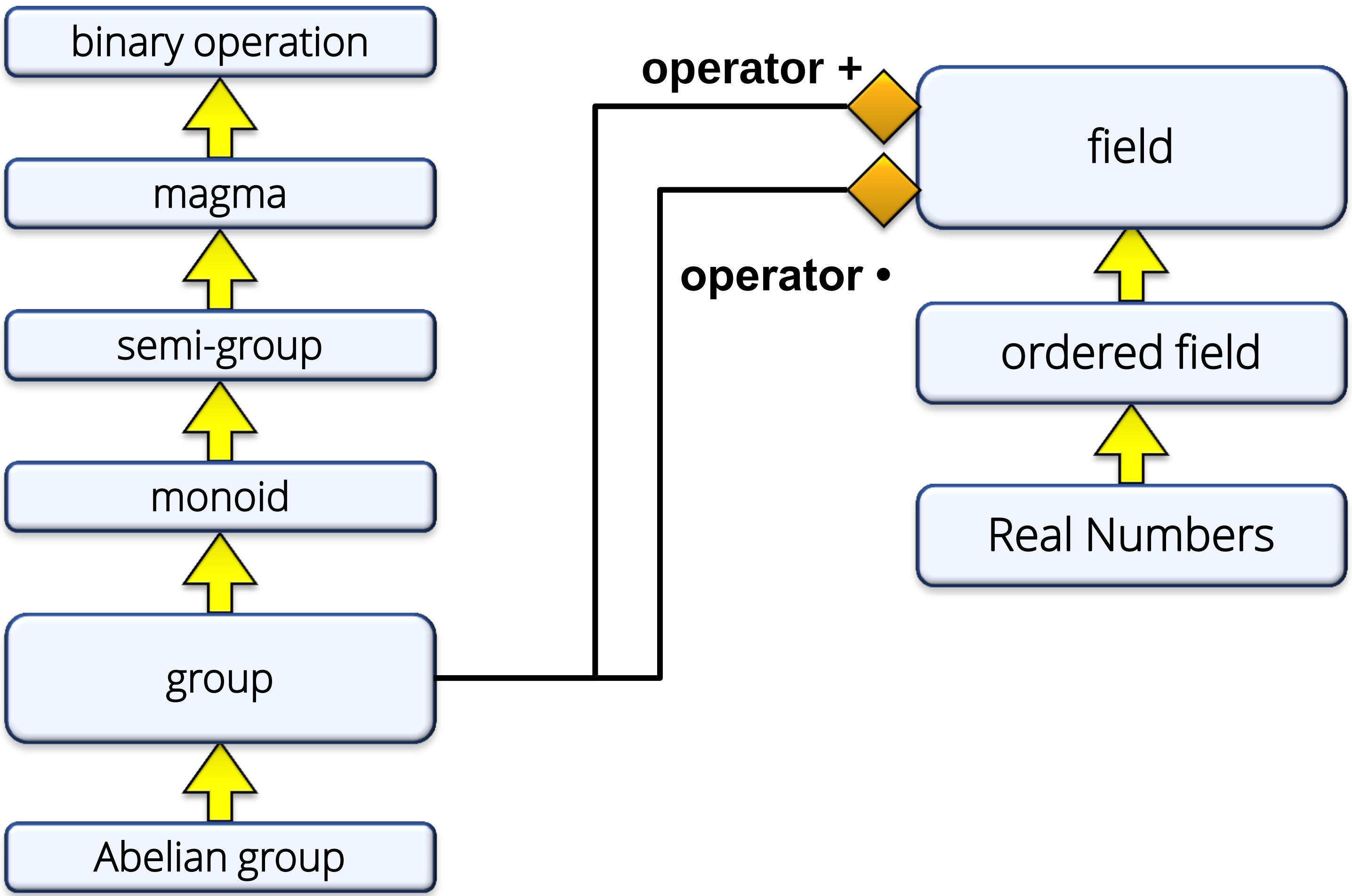
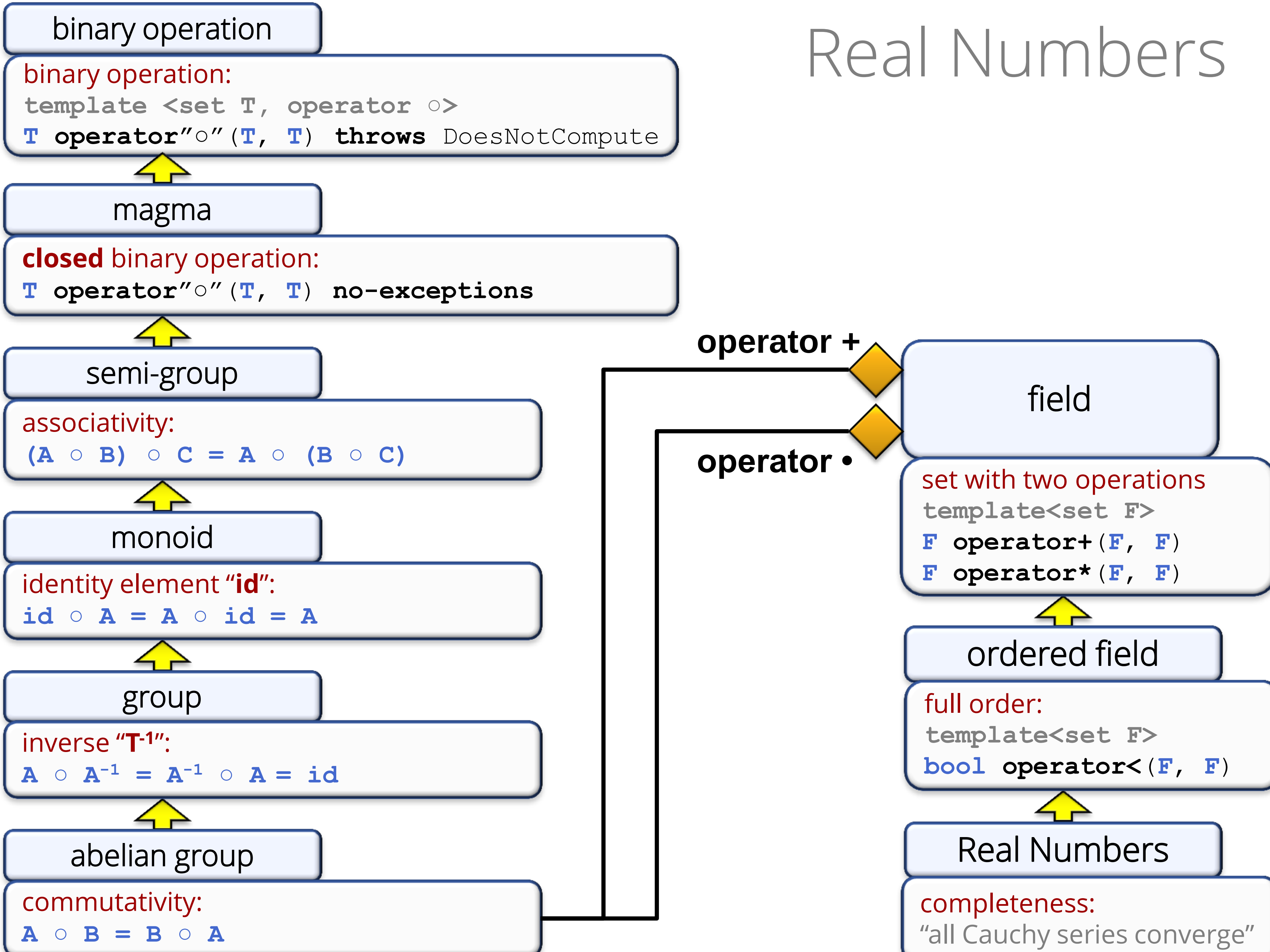
Supplementary



“Class Diagram”
for Real Numbers

Real Numbers



Structure: Vector Space

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advanced topics
main ideas

Vector Spaces

Vector space:

- Set of vectors V
- Based on field $\mathbb{F}$ (we use only $\mathbb{F} = \mathbb{R}$)
- Two operations:
  - Adding vectors $\mathbf{u} = \mathbf{v} + \mathbf{w}$ $(\mathbf{u}, \mathbf{v}, \mathbf{w} \in V)$
  - Scaling vectors $w = \lambda v$ $(\mathbf{u} \in V, \boldsymbol{\lambda} \in \mathbf{F})$

Vector Spaces

Settings
V: vector space
F: field (e.g., $\mathbb{R}$)

Vector space axioms:

- Vector addition – Abelian group:
  - $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $\forall \mathbf{u}, \mathbf{v} \in V$: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $\exists \mathbf{0} \in V: \forall \mathbf{v} \in V$: $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  - $\forall \mathbf{v} \in V: \exists "{-\mathbf{v}}" \in V$: $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
- Compatibility with scalar multiplication:
  - $\forall \mathbf{v} \in V, \lambda, \mu \in F$: $\lambda(\mu \mathbf{u}) = \lambda\mu(\mathbf{u})$
  - $\forall \mathbf{v} \in V$: $1 \cdot \mathbf{v} = \mathbf{v}$
  - $\forall \mathbf{v}, \mathbf{w} \in V, \lambda \in F$: $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$
  - $\forall \mathbf{v} \in V, \lambda, \mu \in F$: $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

Properties

Some differences to our definition

- Abstract vector spaces can have infinite dimension
  - For example: The set of all functions
    $$f: \mathbb{R} \to \mathbb{R}$$
    forms an ∞-dimensional vector space
  - But they always have a basis
    $\to$ coordinate representation
- We can use other fields than $\mathbb{R}$, such as $\mathbb{C}$ or finite fields such as ($\mathbb{Z} \bmod p$, p prime)
- We can recognize them before we have a coordinate representation

Theorem

Theorem ("Basis-Isomorphism")

- Any finite-dimensional vector space can be represented by columns of numbers
  - Use the d coordinates of the d basis vectors (dim= d)

Our definition makes sense

- Special case

Structure: Scalar Product

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Scalar Product

Aximatic Definition: Scalar Product

- Function
  - two vector arguments (input)
  - one scalar output
  - $b: V \times V \rightarrow F$
    - think b == "$operator \circ$"
  - V is a vector space, F is a field (such as $\mathbb{R}$)

Settings
V: vector space
F: field (e.g., $\mathbb{R}$)

Axiomatic Definition: Scalar Product

Settings

$\lambda \in F$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

Properties

- Symmetry

$$b(\mathbf{u}, \mathbf{v}) = b(\mathbf{v}, \mathbf{u})$$

- Bilinearity

$$b(\mathbf{u} + \lambda \mathbf{v}, \mathbf{w}) = b(\mathbf{u}, \mathbf{w}) + b(\lambda \mathbf{v}, \mathbf{w})$$

(linearity in second argument follows from symmetry)

- Positive definite

$$b(\mathbf{u}, \mathbf{u}) \geq 0, \qquad [b(\mathbf{u}, \mathbf{u}) = \mathbf{0}] \Rightarrow [\boldsymbol{u} = \mathbf{0}]$$

Symmetric, positive-definite, bilinear function

General Scalar Product

Theorem

- In a finite-dimensional vector space, any scalar product has the following form:

$$b(\mathbf{x}, \mathbf{y}) = (\mathbf{M}\mathbf{x}) \cdot (\mathbf{M}\mathbf{y}) = \mathbf{x}^{\mathrm{T}}\left(\mathbf{M}^{\mathrm{T}}\mathbf{M}\right)\mathbf{y}$$

  - " $\cdot$ " is the standard scalar product as we defined it
  - **M** is a square matrix with linearly-independent columns
    - I.e., **M** transforms to a different coordinate frame

Our definition still makes sense…

- Special case: undistorted coordinates
- General scalar products can take non-standard coordinate frames into account

Structure: Linear Map

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main ideas

Definition of Linear Maps

Settings
$\mathbf{A}$ – linear map
$\mathbf{v} \in V_1$ - vector

Axioms

- Linear Map: A function

$$\mathbf{A}: V_1 \rightarrow V_2$$

  maps from one vector space (V_1) to another (V_2)

- Linearity requires

$$\mathbf{A}(\mathbf{v} + \mathbf{w}) = \mathbf{A} \cdot \mathbf{v} + \mathbf{A} \cdot \mathbf{w}$$
$$\mathbf{A} \cdot (\lambda \cdot \mathbf{v}) = \lambda \cdot (\mathbf{A} \cdot \mathbf{v})$$

Theorem

- Linear maps in finite-dimensional vector spaces can always be represented by matrices
- Our definition makes sense: special case