T4 answer sheet

1a.

b. $P_{xy} = P_x\overrightarrow{b_1} + P_y\overrightarrow{b_2}$

$P_{uv} = P_u\vec{u} + P_v\vec{v}$ (or: $P_{uv} = E + P_u\vec{u} + P_v\vec{v}$, but we will ignore the translate for now.)

c. Camera matrix: $M = \begin{vmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{vmatrix}$

Inverse camera matrix $M' = M^{-1} = M^T = \begin{vmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{vmatrix}$

d. $P = M'P_{xy} = \begin{vmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{vmatrix} \binom{3}{1} = \begin{pmatrix} \frac{1}{2}\sqrt{2} * 3 + \frac{1}{2}\sqrt{2} * 1 \\ -\frac{1}{2}\sqrt{2} * 3 + \frac{1}{2}\sqrt{2} * 1 \end{pmatrix} = \binom{2\sqrt{2}}{-\sqrt{2}}$.

2a. For rotation in 3D, it didn't matter how we mapped the coordinate systems to each other. Here, it does matter. We want to map the world to camera coordinates in a way that two axes are parallel to the width and height, respectively, of the viewing plane and the third one is orthogonal to it. This is achieved by using the up vector instead of a random one. The up vector is defined as a vector in the plane bisecting the viewer's head into left and right halves and "pointing to the sky".

b. If we are using the negative view vector, our camera points in negative Z-direction. We get the first other axis using the cross product of the up vector with the view vector. Building the cross product of the resulting vector with the view vector gives us the 3rd axis of our camera coordinate system. But how can we control if we end up with a left or right handed one? It depends on the order in which we multiply the vectors in the cross product, since $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

c. An orthonormal basis is a basis where the base vectors are unit vectors that are orthogonal to each other. In graphics, it is very often convenient (and easier) to calculate with normalized vectors. In this particular case, it enables us to very easily determine the rotation matrix needed for the camera transformation.

3a. The orthographic view volume is an axis parallel box defined by its enclosing planes $[l, r] \times [b, t] \times [f, n]$ (see book, page 144-145).

b. The canonical view volume is the cube containing all 3D points whose Cartesian coordinates are between -1 and +1 (see book, page 143).

c. To prove this, we first have to specify the corners of the two volumes. E.g., for the orthographic view volume, one of them is (r, t, n) (which represents the corner on the right and top on the near plane). The corresponding corner on the canonical view volume is (1, 1, 1). Multiplying (r, t, n) with the given matrix shows that it does indeed realize this mapping.

4a. Because we need to be able to map a coodinate to a new value that is created by dividing through the value of another coordinate (e.g. for the x-coordinate: $x_s = \frac{dx}{z}$) which can't be done with matrix multiplication.

b. To prove this, we basically just have to write down the matrices (with generic values for the arbitrary rows) and do the related arithmetic operations to see that it does indeed come to the same result.
Alternatively (and with less writing) you could show this using distributivity of scalar multiplication. Let M be a 4×4 matrix as described, then we have $M(x, y, z, 1) = (x', y', z', 1)$. Then, $M(hx, hy, hz, h) = hM(x, y, z, 1) = h(x', y', z', 1) = (hx', hy', hz', h)$. We see that after homogenization they both map to (x',y',z').

c. See slides.

5. The view vector is the vector specifying the looking direction. In this case, it is $\vec{V} = (3,4,12)$.

6. First, we need to construct an orthonormal basis $(\vec{u}, \vec{v}, \vec{w})$ for the camera. Here, $\vec{w}$ is simply the normalized opposite view vector (note that we look into the negative $\vec{w}$-direction), so $\vec{w} = -\vec{V}/||\vec{V}||$. Filling in the numbers gives $\vec{w} = (3/13,4/13,12/13)$.
The vector $\vec{u}$ is perpendicular to the plane spanned by $\vec{w}$ and the up vector. So we take the cross product of these two vectors, and normalize: $\vec{u} = \overrightarrow{up} \times \vec{w}/(||\overrightarrow{up} \times \vec{w}||)$. In our concrete case we get: $\vec{u} = \left(\frac{4}{\sqrt{17}}, 0, \frac{-1}{\sqrt{17}}\right)$.
Finally, $\vec{v}$ is perpendicular to $\vec{w}$ and $\vec{u}$, so $\vec{v} = \vec{w} \times \vec{u}$. In our case, this gives
$\vec{v} = (\frac{-4}{13\sqrt{17}}, \frac{51}{13\sqrt{17}}, \frac{-16}{13\sqrt{17}})$.
We find the matrix M_{cam} by first translating over $(-x_e, -y_e, -z_e)$, and next multiplying by the matrix where the rows are formed by $\vec{u}$, $\vec{v}$, and $\vec{w}$ (completed with zeros and ones in the appropriate places). The latter matrix aligns the camera coordinate system with the global world coordinate system. If you work out the numbers, you should end up with the matrix:

$$\begin{pmatrix} \frac{4}{\sqrt{17}} & 0 & \frac{-1}{\sqrt{17}} & \frac{-10}{\sqrt{17}} \\ \frac{-4}{13\sqrt{17}} & \frac{51}{13\sqrt{17}} & \frac{-16}{13\sqrt{17}} & \frac{-500}{13\sqrt{17}} \\ \frac{3}{13} & \frac{4}{13} & \frac{12}{13} & \frac{-470}{13} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

7. This matrix can be found in the lecture slides, and in the textbook on page 152. Filling in the numbers is left as an exercise for the reader.

8. See exercise 3. You can also find it in the slides, and in the textbook on page 145.
Completing the answer is a matter of filling in the numbers and doing the matrix multiplications. Notice however, that you are strongly advised to not just copy the formulas and fill in the numbers, but to make sure you understand how we got to this matrix in the first place!

9. Anyone brave enough to compute all matrix multiplications such that the resulting matrix maps the point (7,16,18) onto the center of the image deserves the utmost respect. I didn't do it myself; the result is not very important, but understanding the whole procedure is.