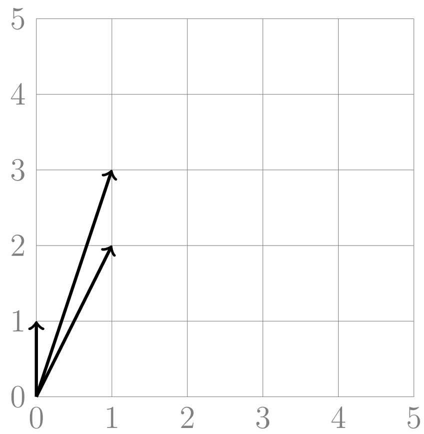
Exercises

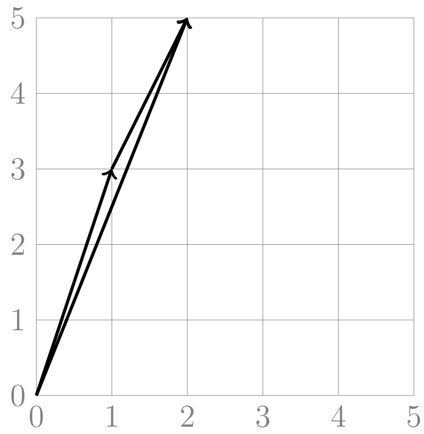
Exercise 1

1a)

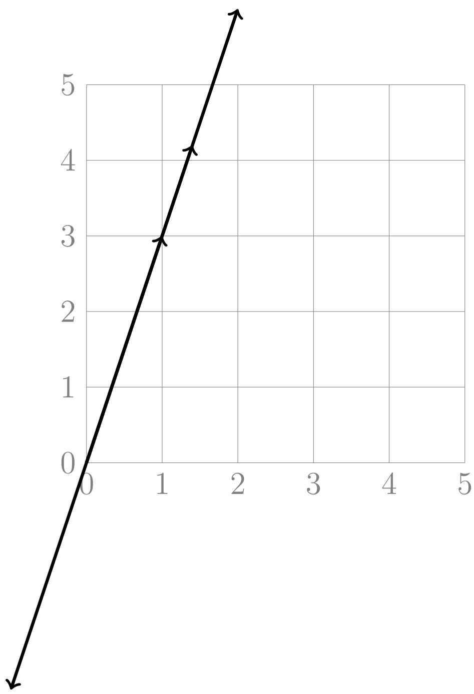


1b)

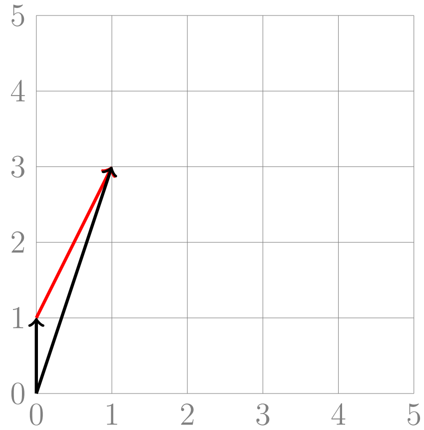
$\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$



1c)



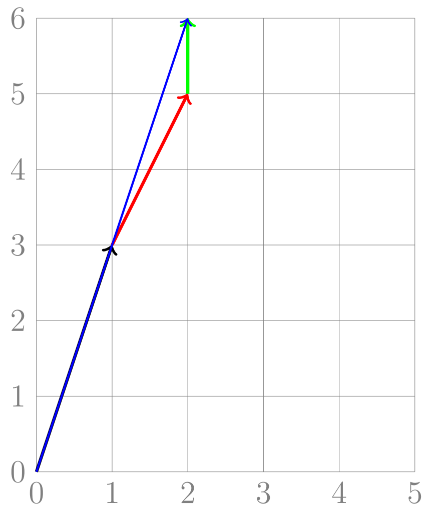
1d)



1e)

$2(\vec{a}+\vec{c}) = 2(\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}) = 2\begin{pmatrix}1\\4\end{pmatrix} = \begin{pmatrix}2\\8\end{pmatrix}$

1f) A linear combination is a new vector made from other vectors, we can choose whatever lambdas we want. $\lambda_1 = \lambda_2 = \lambda_3 = 1$



Exercise 2

$\vec{a} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \vec{b} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \vec{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

2a

$\vec{a} + \vec{b} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$

2b

$|\vec{a}| = \sqrt{1 + 5^2 + 4^2} = \sqrt{1 + 25 + 16} = \sqrt{42}$

2c

$\vec{u} = \frac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$

2d

$\vec{a} \cdot \vec{b} = 1 \cdot 2 + 5 \cdot -5 + 4 \cdot 1 = 2 - 25 + 4 = -19$

2e

The cross product can be skipped, but for whoever cares, we are going to use a method that looks like calculating the determinant. With i, j and k being unit vectors along the x, y and z-axis.

$$\begin{bmatrix} i & 1 & 1 \\ j & 5 & 2 \\ k & 4 & 1 \end{bmatrix} \Rightarrow i = 5 \cdot 1 - 4 \cdot 2, j = 4 \cdot 1 - 1 \cdot 1, k = 1 \cdot 2 - 5 \cdot 1$$

Cross product: $\begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$

Fun fact, the cross product of two vectors a normal vector for both lines.

2f

We create a normalized vector from $\vec{b}$, and multiply this by 2. $|\vec{b}| = \sqrt{4 + 25 + 1} = \sqrt{30} \Rightarrow \vec{p} = \frac{2}{\sqrt{30}} \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$

2g

A 3 dimensional vector has an infinite amount of normal vectors, in this case we can just choose one.

$\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \cdot \vec{n} = 0 \Rightarrow 2a - 5b + c = 0$, neem $a = 1, b = 1, c = 3$. $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ is a normal vector. Note that this equation has a freedom of the second degree, which means that two variables of a, b and c can be freely chosen while the last is calculated.

2h

Remember the funfact given in 2e? We use that here.

$\vec{b} \times \vec{c} = \begin{pmatrix} -7 \\ -1 \\ 9 \end{pmatrix}$

But if you do not want to use the cross-product....

Using the property of the dot-product. $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \cdot \vec{n} = 0$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \vec{n} = 0$,

$$2n_1 - 5n_2 + n_3 = 0 \tag{1}$$

$$n_1 + 2n_2 + n_3 = 0 \tag{2}$$

Thanks to the linear property of these equations we can subtract the second equation twice from the first equation. And we get:

$$-9n_2 - n_3 = 0 \Leftrightarrow n_3 = -9n_2$$

Substitute n_3 in equation 1 and 2.

$$2n_1 - 5n_2 - 9n_2 = 2n_1 - 14n_2 = 0 \tag{3}$$

$$n_1 + 2n_2 - 9n_2 = n_1 - 7n_2 = 0 \tag{4}$$

We can see that $n_1 = 7n_2$. Now we choose $n_2 = -1$, $n_1 = 7n_2 = -7$ and $n_3 = -9n_2 = 9$. So our normal vector is $\begin{pmatrix} -7 \\ -1 \\ 9 \end{pmatrix}$

Exercise 3

The dot product of a vector with itself is the magnitude squared

Exercise 4

In this exercise we use the property of the dot product: $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\alpha)$. Due to $\vec{a}$ and $\vec{b}$ being unit vectors, their length is 1. The property becomes, $\vec{a} \cdot \vec{b} = \cos(\alpha)$

4a

$\vec{a} \cdot \vec{b} = \cos(\frac{\pi}{2}) = \cos(\frac{3\pi}{2}) = 0$, this means that vectors are perpendicular.

4b

We know that the cosine is smaller than 0 when the radius is larger than 90 degrees.

4c

$\vec{a} \cdot \vec{b} = \cos(0) = \cos(2\pi) = 1$, so the angle is 0 degrees.

Exercise 5

5a

It's not 1 or -1, since both vectors can not be parallel. Assuming the vectors are unit vectors.

5b

It's 1 because $\vec{b}$ is a unit vector, since an orthonormal basis is made out set of vectors that are all perpendicular to each other (orthogonal) and where each vector is a unit vector.

5c

$\vec{b} = \frac{1}{2}\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$, or $\vec{b} = \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}$

5d

$\vec{b} = \frac{1}{2}\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$

$$\lambda_1 \cdot \frac{1}{2}\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} + \lambda_2 \cdot \frac{1}{2}\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \lambda_1 = \frac{3}{\sqrt{2}}, \lambda_2 = -\frac{1}{\sqrt{2}}$$

Exercise 6

6a

$f(x) = \frac{2}{5}x + \frac{6}{5}$

6b

$f(-3) = 0, f(2) = 2$

6c

$\frac{2}{5}x - y = -\frac{6}{5} \Leftrightarrow 2x - 5y + 6 = 0$

6d

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \lambda_1 + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

2 normals, $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

6e

There is no relation, C only moves the line around but does not change direction.

Exercise 7

This exercise is simply the definition of the dot product; hence left out of the solution.

Exercise 8

Parametric: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \lambda + \begin{pmatrix} c \\ d \end{pmatrix}$ or more general: $\vec{P} = \vec{r}\lambda + \vec{r}_0$
Slope-intersect: $y = mx + c$
Implicit: $Ax + By + C = 0$

Exercise 9

We can solve this following how it was done in class. Alternatively, we can do it as follows.

We start by taking a few definitions. First consider a line for which $b \neq 0$, i.e., the line is not parallel to the y-axis. Then define a small number $\epsilon > 0$, and take point $P = (x, y)$ for which $ax + by + c = 0$. Also, b can not be 0.

For the positive half we take a new point $P_+ = (x, y + \epsilon)$ and show that this is larger than 0.

$$f(x, y + \epsilon) = ax + b(y + \epsilon) + c = ax + by + c + b\epsilon > ax + by + c = 0$$

Therefore, for a point p in the positive half, $f(x_p, y_p) > 0$.

We do the same for the point in the negative half, we take point $P_-(x, y - \epsilon)$ and show that this is less than 0.

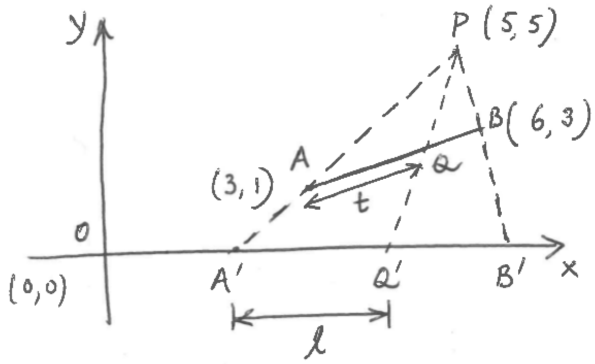
$$f(x, y + \epsilon) = ax + b(y - \epsilon) + c = ax + by + c - b\epsilon < ax + by + c = 0$$

Therefore, for a point P_- in the negative half, $f(x_p, y_p) < 0$.

The above method does not work if $b = 0$. However, note that $b = 0$ corresponds to the line being parallel to the y-axis. In that case on the right of the line $f(x, y) > 0$ and on the left it is < 0.

In other words, the line $f(x, y) = 0$ divides the two-dimensional plane in two parts. In one of them $f(x, y) > 0$, and in the other $f(x, y) < 0$.

Exercise 10



Note that here we do not use unit vectors for the parametric equations for the lines.

10a

Finding A', we create a parametric line from P to A.

$$\begin{pmatrix}5\\5\end{pmatrix} + \left(\begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}5\\5\end{pmatrix}\right)\lambda_1 = \begin{pmatrix}5\\5\end{pmatrix} + \begin{pmatrix}-2\\-4\end{pmatrix}\lambda_1 \tag{5}$$

We use equation 5 to find a point A' on the x-axis and we know that $A' = (x_a, 0)$.

$$\begin{pmatrix}5\\5\end{pmatrix} + \begin{pmatrix}-2\\-4\end{pmatrix}\lambda_1 = \begin{pmatrix}x_a\\0\end{pmatrix}$$

We find that $\lambda_1 = \frac{5}{4}$, and therefore $x_a = 5 - 2 \cdot \frac{5}{4} = \frac{5}{2}$. So $A' = (\frac{5}{2}, 0)$

Same for B'. The parametric line from P to B.

$$\begin{pmatrix}5\\5\end{pmatrix} + \left(\begin{pmatrix}6\\3\end{pmatrix} - \begin{pmatrix}5\\5\end{pmatrix}\right)\lambda_2 = \begin{pmatrix}5\\5\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix}\lambda_2 \tag{6}$$

We use equation 6 to find a point B' on the x-axis and we know that $B' = (x_b, 0)$.

$$\begin{pmatrix}5\\5\end{pmatrix} + \begin{pmatrix}1\\-2\end{pmatrix}\lambda_2 = \begin{pmatrix}x_b\\0\end{pmatrix}$$

We find that $\lambda_2 = \frac{2}{5}$, and therefore $x_a = 5 + 1 \cdot \frac{2}{5} = \frac{27}{5}$. So $B' = (\frac{27}{5}, 0)$.

10b

Since point Q is depending on t, we need to express point Q as a function of t.

We make a line from A to B.

$$\begin{pmatrix}3\\1\end{pmatrix} + \left(\begin{pmatrix}6\\3\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix}\right)\lambda = \begin{pmatrix}3\\1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}\lambda$$

To find point Q we normalize the direction vector in equation so that we can use constant t instead of the λ. Point Q will then be placed at:

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} + \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}t$$

Now that we found point Q, we need to find point Q' on the x-axis. We do this in the same way as in 10a, we construct the line from P to Q in a parametric form and use that to calculate where point Q' is.

$$\begin{pmatrix}5\\5\end{pmatrix} + \left(\begin{pmatrix}3\\1\end{pmatrix} + \frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}t - \begin{pmatrix}5\\5\end{pmatrix}\right)\lambda = \begin{pmatrix}5\\5\end{pmatrix} + \left(\frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}t - \begin{pmatrix}2\\4\end{pmatrix}\right)\lambda \quad (7)$$

Since we know that $Q' = (x_q, 0)$, we can use equation 7 to solve x_q.

$$\begin{pmatrix}5\\5\end{pmatrix} + \left(\frac{1}{\sqrt{13}}\begin{pmatrix}3\\2\end{pmatrix}t - \begin{pmatrix}2\\4\end{pmatrix}\right)\lambda = \begin{pmatrix}x_q\\0\end{pmatrix}$$

We find that $\lambda = \frac{-5}{\frac{2t}{\sqrt{13}} - 4}$, and therefore $x_q = 5 + \left(\frac{3t}{\sqrt{13}} - 2\right) \cdot \left(\frac{-5}{\frac{2t}{\sqrt{13}} - 4}\right)$.
In conclusion, we can express the length of l as a function of t:

$$l = 5 + \left(\frac{3t}{\sqrt{13}} - 2\right) \cdot \left(\frac{-5}{\frac{2t}{\sqrt{13}} - 4}\right) - \frac{5}{2}$$