Tutorial 2, exe

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Exercise 1

1a

$(x-5)^2 + (y-1)^2 = r^2 = 4^2 = 16$

1b

$y = 1 + r \cdot \sin\theta$, $x = 5 + r \cdot \cos\theta$

1c

$(1 + 4\sin\frac{\pi}{4}, 5 + 4\cos\frac{\pi}{4}) = (1 + 4\sqrt{2}, 5 + 4\sqrt{2})$. The radial unit vector is then: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

1d

$(1 + 4\sin\frac{3\pi}{4}, 5 + 4\cos\frac{3\pi}{4}) = (1 + 4\sqrt{2}, 5 - 4\sqrt{2})$. The radial unit vector is then: $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Exercise 2

2a

$$x_p = \sqrt{2}\cos\theta_1 + \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

$$y_p = \sqrt{2}\sin\theta_1 + \sqrt{2}$$

$$x_q = \sqrt{2}\cos\theta_2 + \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$$

$$y_q = \sqrt{2}\sin\theta_2 + \sqrt{2}$$

2b

To find the tangent of point P, we need to take the normal of the line CP, where C is the centre. The direction vector of this line is:

$$\begin{pmatrix} \sqrt{2}\cos\theta_1 + \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \\ \sqrt{2}\sin\theta_1 + \sqrt{2} \end{pmatrix} - \begin{pmatrix} \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2}\cos\theta_1 \\ \sqrt{2}\sin\theta_1 \end{pmatrix}$$

The unit normal of this line:

$$\begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix}$$

So the tangent of point P is then:

$$\begin{pmatrix} \sqrt{2}\cos\theta_1 + \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \\ \sqrt{2}\sin\theta_1 + \sqrt{2} \end{pmatrix} + \begin{pmatrix} \sin\theta_1 \\ -\cos\theta_1 \end{pmatrix} \lambda,$$

In the same way, we find that the unit normal for point CQ is:

$$\begin{pmatrix} \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix}$$

And the tanget:

$$\begin{pmatrix} \sqrt{2}\cos\theta_2 + \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}} \\ \sqrt{2}\sin\theta_2 + \sqrt{2} \end{pmatrix} + \begin{pmatrix} \sin\theta_2 \\ -\cos\theta_2 \end{pmatrix} \lambda$$

For specific values of θ_1 and θ_2. and correspondingly, of λ (you can calculate these using trigonometry as discussed in mathematics lecture 1), the lines OP and OQ are going to be tangents to the circle at P and at Q respectively. Note here the advantage of using unit tangent vectors: then $|\lambda|$ is simply the length of the lines OP and OQ. In other words, the part of the circle that the eye is able to see is confined between P and Q.

Exercise 3

3a

We have two points that we can see as vectors, we create a vector from one to the other. $\begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix}$ We can now calculate the length of this vector. $\sqrt{6^2 + 1^2 + 4^2} = \sqrt{53} = 7.28$

3b

$$\vec{p}(t) = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix} \lambda$$

3c

$\begin{pmatrix}-1\\6\\0\end{pmatrix}$ (swapping x/-y, zeroing z) and $\begin{pmatrix}-4\\0\\6\end{pmatrix}$ (swapping x/-z, zeroing y)

3d

Unlimited, they lie on planes perpendicular to this straight line.

3e

Plane: $\begin{pmatrix}-1\\6\\0\end{pmatrix}\lambda_1 + \begin{pmatrix}-4\\0\\6\end{pmatrix}\lambda_2 + \vec{c}$.

For the implicit form we can use: $6x + 1y + 4z + d = 0$

Exercise 4

4a

The line vector $\begin{pmatrix}6\\1\\4\\-2\end{pmatrix}$ Two normals $\begin{pmatrix}-1\\6\\0\\0\end{pmatrix}$ (swapping x/-y, zeroing y and z) and $\begin{pmatrix}0\\0\\2\\4\end{pmatrix}$ (swapping z/-w, zeroing x and y).

4b

Unlimited, in a sphere of radius 1 around the 4D line.

Exercise 5

5a

The plane is perpendicular to the x-axis; so all vectors on this plane has to have zero x-components. Two linearly independent vectors on a plane with $x = 1$: $\begin{pmatrix}0\\9\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\42\end{pmatrix}$. [Any two vectors with zero x-components that are not (anti-)parallel to each other will do.]

5b

We could calculate the cross product of the two vectors we just found, but we don't have to since we simply have $\begin{pmatrix}1\\0\\0\end{pmatrix}$ pointing to the x-direction; i.e., perpendicular to the plane.

5c

The closest distance from point P (5, 1, 1) to the plane is via the normal vector of said plane. We can see this distance is 4 but we will show how to calculate it anyway.

We need to find a point P' on the plane that is closest to P. We do this by finding the intersection of point P to the plane. We can find this intersection by:

$$P^T + \begin{pmatrix}1\\0\\0\end{pmatrix}\lambda = \begin{pmatrix}5\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}\lambda = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\9\\1\end{pmatrix}\lambda_1 + \begin{pmatrix}0\\1\\42\end{pmatrix}\lambda_2$$

We get three equations:

$$5 + \lambda = 1$$
$$1 = 9\lambda_1 + \lambda_2$$
$$1 = \lambda_1 + 42\lambda_2$$

Solving these equations we find that $\lambda = -4$, $\lambda_2 = \frac{8}{377}$ and $\lambda_1 = \frac{369}{377}$. Only λ is important now.

Since we found for what λ the intersection exists, we can use the direction vector multiplied with λ to find the length.

$$\left| -4 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} \right| = \sqrt{(-4)^2 + 0^2 + 0^2} = 4$$

5d

There are infinite planes with distance 0 to the origin, it's all the planes rotating around the x-axis.

Exercise 6

6a

The first vector to create a plane: $\vec{p}_1 - \vec{p}_2 = \begin{pmatrix}0\\7\\6\end{pmatrix} - \begin{pmatrix}8\\0\\8\end{pmatrix} = \begin{pmatrix}-8\\7\\-2\end{pmatrix}$. The second vector to create a plane $\vec{p}_1 - \vec{p}_3 = \begin{pmatrix}0\\7\\6\end{pmatrix} - \begin{pmatrix}12\\10\\0\end{pmatrix} = \begin{pmatrix}-12\\-3\\6\end{pmatrix}$. This makes the parametric equation:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\7\\6\end{pmatrix} + \begin{pmatrix}-8\\7\\-2\end{pmatrix}\lambda_1 + \begin{pmatrix}-12\\-3\\6\end{pmatrix}\lambda_2$$

By varying λ_1 and λ_2 we can find all points on the plane.

6b

We can use a normal vector from the plane to find the implicit equation. So let s find the normal vector of this plane.

A normal vector should be perpendicular on both directional vectors in the plane. By using the dot product we get two equations,

$$\begin{pmatrix}-8\\7\\-2\end{pmatrix} \cdot \vec{n} = -8n_1 + 7n_2 - 2n_3 = 0 \quad (1)$$

$$\begin{pmatrix}-12\\-3\\6\end{pmatrix} \cdot \vec{n} = -12n_1 - 3n_2 + 6n_3 = 0 \quad (2)$$

We can use the linear property of these equations to make a new equation with one variable less. We add three of equation 1 to equation 2 to create a new equation.

$$-36n_1 + 18n_2 = 0 \Leftrightarrow n_2 = 2n_1 \quad (3)$$

Putting this in equation 1, we get:

$$-8n_1 + 14n_1 - 2n_3 = 6n_1 - 2n_3 = 0 \Leftrightarrow n_3 = 3n_1 \quad (4)$$

Let's choose $n_1 = 1$, then it follows that $n_2 = 2$ and $n_3 = 3$.

Now that we have a normal vector, the implicit equation of the plane has a form of:

$$1x + 2y + 3z = C$$

to find C, we fill in a point on the plane. Let's take p_1. We get:

$$1x + 2y + 3z = 1 \cdot 0 + 2 \cdot 7 + 3 \cdot 6 = 32$$

6c

To verify if the points are indeed on the plane, we fill in the points coordinates in the equation and check if this equals 32. We already did this for point p_1. For p_2: $1 \cdot 8 + 2 \cdot 0 + 3 \cdot 8 = 32$ and for p_3: $1 \cdot 12 + 2 \cdot 10 + 3 \cdot 0 = 32$.

6d

Look at exercise 6b.

Exercise 7

7a

Parametric equation of l_1

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\7\\6\end{pmatrix} + \left(\begin{pmatrix}0\\7\\6\end{pmatrix} - \begin{pmatrix}8\\0\\8\end{pmatrix}\right)\lambda = \begin{pmatrix}0\\7\\6\end{pmatrix} + \begin{pmatrix}-8\\7\\-2\end{pmatrix}\lambda$$

7b

Parametric equation of l_2

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 12 \\ 10 \\ 0 \end{pmatrix} \right) \lambda = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} + \begin{pmatrix} -12 \\ -3 \\ 6 \end{pmatrix} \lambda$$

7c

The intersection point of l_1 and l_2 is point p_1. Due to the fact they both lines start from the same point. This is also the only intersection point because they are lines that are not parallel to each other. If the lines were parallel then there would be an infinite intersection points.

Exercise 8

8a

To display the x_{world} on x_{screen}, we need to translate the x_w to the right, and then scale it. We could it as a line from -2 to 2 along the x-axis. This line needs to be moved and scaled to span the line from 0 to 1023. $x_{\text{screen}} = (2 + x_{\text{world}}) \cdot \frac{1023}{4}$.

We do the same for y_{world} to be displayed on y_{screen}. Except this time we mirror it along the x-axis. So that the positive part of the y-axis is down, and the negative is up. $y_{\text{screen}} = -(-1.5 + y_{\text{world}}) \cdot \frac{767}{3}$.

8b

Let:

$$f \begin{pmatrix} x_{\text{world}} \\ y_{\text{world}} \end{pmatrix} = \begin{pmatrix} x_{\text{screen}} \\ y_{\text{screen}} \end{pmatrix} = \begin{pmatrix} (2 + x_{\text{world}}) \cdot \frac{1023}{4} \\ -(-1.5 + y_{\text{world}}) \cdot \frac{767}{3} \end{pmatrix}$$

We can fill the four points into function f.

Top left:

$$f \begin{pmatrix} -2 \\ 1.5 \end{pmatrix} = \begin{pmatrix} (2 - 2) \cdot \frac{1023}{4} \\ -(-1.5 + 1.5) \cdot \frac{767}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Top right:

$$f \begin{pmatrix} 2 \\ 1.5 \end{pmatrix} = \begin{pmatrix} (2 + 2) \cdot \frac{1023}{4} \\ -(-1.5 + 1.5) \cdot \frac{767}{3} \end{pmatrix} = \begin{pmatrix} 1023 \\ 0 \end{pmatrix}$$

Bottom right:

$$f \begin{pmatrix} 2 \\ -1.5 \end{pmatrix} = \begin{pmatrix} (2 + 2) \cdot \frac{1023}{4} \\ -(-1.5 - 1.5) \cdot \frac{767}{3} \end{pmatrix} = \begin{pmatrix} 1023 \\ 767 \end{pmatrix}$$

Bottom left:

$$f \begin{pmatrix} -2 \\ -1.5 \end{pmatrix} = \begin{pmatrix} (2 - 2) \cdot \frac{1023}{4} \\ -(-1.5 - 1.5) \cdot \frac{767}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 767 \end{pmatrix}$$

8c

World origin (0, 0) should appear on the middle on the screen.

$$f\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}(2+0)\cdot\frac{1023}{4}\\-(-1.5+0)\cdot\frac{767}{3}\end{pmatrix} = \begin{pmatrix}511.5\\383.5\end{pmatrix}$$

Which is the center of the screen.

8d

$$x_{\text{screen}} = (x_1 + x_{\text{world}})\cdot\frac{W}{x_2 - x_1}$$
$$y_{\text{screen}} = -(-y_1 + y_{\text{world}})\cdot\frac{H}{y_2 - y_1}$$

8e

We have a width W. We shorten this to world coordinates. So $\frac{W}{x_2-x_1}$ and divide this by W to get the width of a pixel in world coordinates. Width is $\frac{1}{x_2-x_1}$

The same goes for the height of a pixel. The height is $\frac{1}{y_2-y_1}$.

8f

Suppose we have a given screen aspect ratio r. Since then $\frac{x}{r} = y$:

$$x_{\text{screen}} = (x_1 + x_{\text{world}})\cdot\frac{W}{x_2 - x_1}$$
$$y_{\text{screen}} = -(-\frac{x_1}{r} + y_{\text{world}})\cdot\frac{rH}{x_2 - x_1}$$

8g

$$x_{\text{screen}} = (x_1 + x_{\text{world}})\frac{W}{x_2 - x_1} \Leftrightarrow x_{\text{screen}}\cdot\frac{x_2 - x_1}{W} - x_1 = x_{\text{world}}$$
$$y_{\text{screen}} = -(-\frac{x_1}{r} + y_{\text{world}})\cdot\frac{rH}{x_2 - x_1} \Leftrightarrow -y_{\text{screen}}\cdot\frac{x_2 - x_1}{rH} + \frac{x_1}{r} = y_{\text{world}}$$