Tutorial 3

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Exercise 1

It is:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

Exercise 2

Finding the cooridnates of A' and B'. We create the lines EA and EB.

$$EA = \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \left(\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} \right) \lambda = \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 10 \end{pmatrix} \lambda$$

We find point A' by intersecting line EA with the $z = 0$ plane.

$$\begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 10 \end{pmatrix} \lambda = \begin{pmatrix} x_{A'} \\ y_{A'} \\ 0 \end{pmatrix}$$

Note that this is actually three equations:

$$\begin{aligned} 4 - \lambda &= x_{A'} \\ 4 &= y_{A'} \\ -5 + 10\lambda &= 0 \end{aligned}$$

We find that $\lambda = \frac{1}{2}$

Thus, $A' = (3.5, 4)$.

We do the same for B', B' is the intersection of line EB and plane $z = 0$.

$$EB = \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \left(\begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} \right) \lambda = \begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 14 \end{pmatrix} \lambda$$

Intersect this line with $z = 0$.

$$\begin{pmatrix} 4 \\ 4 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 14 \end{pmatrix} \lambda = \begin{pmatrix} x_{B'} \\ y_{B'} \\ 0 \end{pmatrix}$$

We find $\lambda = \frac{5}{14}$, so $B' = (4 + \frac{5}{14}, 4 + \frac{20}{14})$. This problem has been treated in symbolic manipulation in the class, and the precise problem has been solved by the TAs. Please follow the scanned lecture notes. The solution for the co-ordinates of the point P' is given by

$$\frac{lz_E[u_z(x_A - x_E) - u_x(z_A - z_E)]}{(z_A - z_E)(z_A + lu_z - z_E)}$$

and

$$\frac{lz_E[u_z(y_A - y_E) - u_y(z_A - z_E)]}{(z_A - z_E)(z_A + lu_z - z_E)},$$

where $A = (x_A, y_A, z_A) = (3, 4, 5)$ and $E = (x_E, y_E, z_E) = (4, 4, -5)$, and

$$\hat{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

is the unit vector along the line AB.

Exercise 3

3a

The parametric equation of the sphere:

$$\begin{aligned} x &= 3 + 3.14 \cdot \cos\theta \sin\phi \\ y &= 5 + 3.14 \cdot \sin\theta \sin\phi \\ z &= 1 + 3.14 \cdot \cos\phi \end{aligned}$$

3b

We know that a full circle is 2π radians. So a half circle is π radians. We can take $\phi = \frac{\pi}{2}$, so the circle is on a flat plane. Then for opposite points we can take $\theta_1 = 2\pi$ and $\theta_2 = \pi$.

Point A gives: $\begin{pmatrix}3\\5\\1\end{pmatrix} + 3.14 \begin{pmatrix}\cos 2\pi \sin\frac{\pi}{2}\\ \sin 2\pi \sin\frac{\pi}{2}\\ \cos\frac{\pi}{2}\end{pmatrix} = \begin{pmatrix}3\\5\\1\end{pmatrix} + 3.14 \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}6.14\\5\\1\end{pmatrix}$

Point B gives: $\begin{pmatrix}3\\5\\1\end{pmatrix} + 3.14 \begin{pmatrix}\cos \pi \sin\frac{\pi}{2}\\ \sin \pi \sin\frac{\pi}{2}\\ \cos\frac{\pi}{2}\end{pmatrix} = \begin{pmatrix}3\\5\\1\end{pmatrix} + 3.14 \begin{pmatrix}-1\\0\\0\end{pmatrix} = \begin{pmatrix}-0.14\\5\\1\end{pmatrix}$

3c

To find the distance between these two points we create a vector between these two points and calculate its length.

$$|A - B| = |\begin{pmatrix}6.14\\5\\1\end{pmatrix} - \begin{pmatrix}-0.14\\5\\1\end{pmatrix}| = |\begin{pmatrix}6.28\\0\\0\end{pmatrix}| = \sqrt{6.28^2 + 0^2 + 0^2} = 6.28$$

The distance between these two points is 6.28.

Exercise 4

4a

The implicit representation:

$$(x-3)^2 + (y-3)^2 + (z-3)^2 = r^2$$

We can fill in the point P $(2, 5, 1)$ in the implicit representation to calculate t^2.

$$(2-3)^2 + (5-3)^2 + (1-3)^2 = (-1)^2 + (-2)^2 + (-2)^2 = 1 + 4 + 4 = 9$$

The final implicit representation is:

$$(x-3)^2 + (y-3)^2 + (z-3)^2 = 3^2$$

4b

The shortest distance from a point to a sphere is the distance from the centre of the sphere to the point minus the radius of the sphere.

The distance from the centre to point Q $(4, 9, -1)$:

$$\left| \begin{pmatrix} 4 \\ 9 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \right| = \left| \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} \right| = \sqrt{1^2 + 6^2 + (-4)^2} = \sqrt{1 + 36 + 16} = \sqrt{53}$$

So the distance from the sphere to point Q is $\sqrt{53} - 3$.

4c

Define a tangent plane on point P.

We are going to define the plane in the form

$$(\vec{p} - \vec{p_0}) \cdot \vec{n} = 0$$

Here: $\vec{n}$ is the normal vector on point P. We know that this is the vector from the centre to point P. We get $\vec{n} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$

Our plane is:

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = 0$$

We can rewrite this to:

$$-(x-2)+2(y-5)-2(z-1) = -x+2+2y-10-2z+2 = -x+2y-2z-6 = 0$$

And finally:

$$-x + 2y - 2z = 6$$

For the parametric form: to find the same plane but in parametric form, we find two perpendicular vectors on the normal of the sphere at point P.

This normal is then given by: $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$.

The two perpendicular vectors: $\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} n_0 \\ n_1 \\ n_2 \end{pmatrix} = -n_0 + 2n_1 - 2n_2 = 0$

With this we can calculate two by choosing values for n_0, n_1 and n_2. vectors: $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$.

And we finally get the plane in parametric form:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\5\\1\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}\lambda_1 + \begin{pmatrix}-2\\0\\1\end{pmatrix}\lambda_2$$

4d

The normal of the sphere at point P: $\begin{pmatrix}1\\2\\-2\end{pmatrix}$ We need to find a vector that is perpendicular to this normal, and then another vector that is perpendicular to the previous two. These three vectors then form an orthogonal basis, which means that every single vectors is perpendicular to all the other vectors.

$$\begin{pmatrix}1\\2\\-2\end{pmatrix} \cdot \begin{pmatrix}n_0\\n_1\\n_2\end{pmatrix} = n_0 + 2n_1 - 2n_1 = 0$$

By filling in n_0, n_1 and n_2 we find some vector $\begin{pmatrix}0\\1\\1\end{pmatrix}$. Now we need to find the last vector where:

$$\begin{pmatrix}1\\2\\-2\end{pmatrix} \cdot \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix} = a_0 + 2a_1 - 2a_2 = 0 \tag{1}$$

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \cdot \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix} = a_1 + a_2 = 0 \tag{2}$$

By adding equation (2) to equation (1) twice we get a equation with less variables:

$$a_0 + 4a_1 = 0$$

Take $a_1 = 1$, we get $a_0 = -4$.

By filling these values in equation (2) we find that $a_2 = -1$.

So the two vectors that we were asked to find are $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$

A set of vectors is orthonormal is the set is orthogonal **and** all vectors are unit vectors.

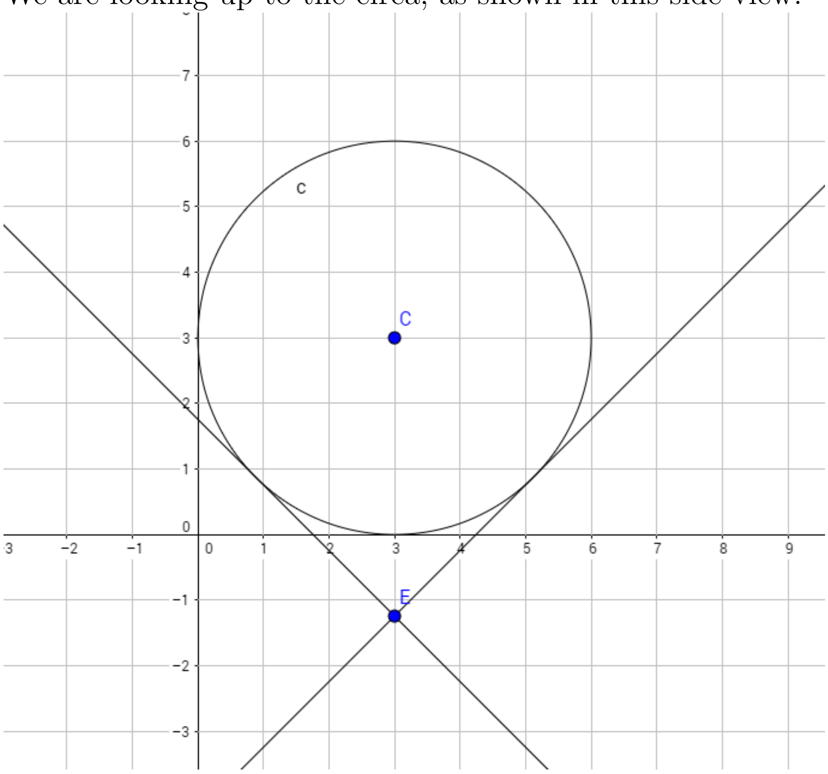
Exercise 5

5a

The parametric representation of the circle

$$x = 3 + 3 \cdot \cos\theta \sin\phi$$
$$y = 3 + 3 \cdot \sin\theta \sin\phi$$
$$z = 3 + 3 \cdot \cos\phi$$

We are looking up to the circa, as shown in this side view:



As you can see we do not see the complete sphere, but only the lower part of it. For the ϕ and θ, we need to calculate the tangents.

To find the tangent lines (though it's actually a tangent cone)

$$\left(\begin{pmatrix}3\\3\\3\end{pmatrix}+3\begin{pmatrix}\cos\theta\sin\phi\\\sin\theta\sin\phi\\\cos\phi\end{pmatrix}-\begin{pmatrix}3\\3\\3-3\sqrt{2}\end{pmatrix}\right)\cdot\left(\begin{pmatrix}3\\3\\3\end{pmatrix}+3\begin{pmatrix}\cos\theta\sin\phi\\\sin\theta\sin\phi\\\cos\phi\end{pmatrix}-\begin{pmatrix}3\\3\\3\end{pmatrix}\right)=0$$

$$\left(\begin{pmatrix}3\cos\theta\sin\phi\\3\sin\theta\sin\phi\\3\cos\phi\end{pmatrix}+\begin{pmatrix}0\\0\\3\sqrt{2}\end{pmatrix}\right)\cdot\left(\begin{pmatrix}3\cos\theta\sin\phi\\3\sin\theta\sin\phi\\3\cos\phi\end{pmatrix}\right)=0$$

We can simplify further, remember that $\cos^2\alpha+\sin^2\alpha=1$:

$$\begin{aligned}9\cos^2\theta\sin^2\phi+9\sin^2\theta\sin^2\phi+(3\cos\phi+3\sqrt{2})\cdot3\cos\phi&=\\9\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)+9\cos^2\phi+9\sqrt{2}\cdot\cos\phi&=\\9(\sin^2\phi+\cos^2\phi)+9\sqrt{2}\cdot\cos\phi&=\\9+9\sqrt{2}\cos\phi&=0\Leftrightarrow\\-9=9\sqrt{2}\cos\phi\Leftrightarrow\cos\phi=-\frac{9}{9\sqrt{2}}&=-\frac{1}{\sqrt{2}}\end{aligned}$$

We have seen that θ can be anything, so we will keep that range $[0,2\pi]$ since it's a sphere. By solving this we find $\phi=2\pi-\frac{3\pi}{4},\phi=\frac{3\pi}{4}$. Remember that the angles of ϕ are measured from the z-axis.

Using the picture we drew, we can conclude that the ranges are:

$$\begin{aligned}&\phi\in\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]\\&\theta\in[0,2\pi]\end{aligned}$$