Antwoorden Graphics Tutorial 4

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1 Exercise 1

- a. $A + A = \begin{bmatrix} 8 & 2 \\ 4 & 2 \\ 6 & 4 \end{bmatrix}$
- b. No you cannot; the height of the second operand must equal the width of the first operand. However, these are not as such (they are 2 and 3)
- c.The transpose is the matrix such that for every element a_{ij} in the original, it is in the transposed matrix a_{ji}. So; $A^T = \begin{bmatrix} 4 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$
- d. $\begin{bmatrix} 17 & 9 & 14 \\ 9 & 5 & 8 \\ 14 & 8 & 13 \end{bmatrix}$

2 Exercise 2

- a. Rule of Sarrus: $|A| = ad \times bc \rightarrow |A| = 1 \times 3 - 2 \times 2 = -1$
- b. $|B| = (1 \times -6) - (2 \times -3) = 0$
- c. Basis vectors of A are counter-clockwise; vectors of B are not independent. Note: the -1 result does not mean the vectors in A are perpendicular; $|A|$ is not orthonormal.

3 Exercise 3

This area is the same as the absolute value of the determinant of matrix

$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}$
$|A| = -14$.
So the area is 14.

4 Exercise 4

a. en b.

$$\det \begin{bmatrix} -2 & 2 & 3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix} = -2 \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} \times -1^2 + 2 \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \times -1^3 + 3 \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} \times -1^4$$

$= -2(1\times-1+3\times0\times-1)\times1+2(-1\times-1+3\times2\times-1)\times-1+3(-1\times0+1\times2\times-1)\times1$

$= -2 \times -1 + 2 \times -5 \times -1 + 3 \times -2 = 6$

5 Exercise 5

- a. 0
- b. Column 3 is column 1 scaled by -2; the three axes do not span a volume but a plane. So it has zero volume.

6 Exercise 6

Here, it is sufficient to provide one case where, if $AB = AC$, $A \neq C$, e.g:
$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$
AB is the null matrix; AC is the null matrix, but $B \neq C$

7 Exercise 7

Write out $A^T A$ in its general form. Now write down what a coefficient in the main diagonal of the resulting matrix would look like. If done correctly, we see that it is the dot product of a column vector of A with itself. Because of orthonormality, this will yield 1.

Now write down the general form of a random coefficient a_{ij} with $i \neq j$. If done correctly, we see that this is the dot product of two non-equal column vectors of the matrix. Since these are perpendicular to each other, the dot product yields 0. Because both the a_{ii} coefficient from the first part, and the a_{ij} coefficient from the second part have been chosen randomly, the proven statements applies for every possible i and i,j-pair (with $i \neq j$). Hence, it follows that the resulting matrix is the identity matrix.

8 Exercise 8

$C_A = \begin{bmatrix} -9 & -1 & -2 \\ 9 & -1 & -2 \\ 18 & -6 & 6 \end{bmatrix}$; $\tilde{A} = C^T = \begin{bmatrix} -9 & 9 & 18 \\ -1 & -1 & -6 \\ -2 & -2 & 6 \end{bmatrix}$; $|A| = -18$;

$A^{-1} = \tilde{A}/|A| = -\frac{1}{18} \begin{bmatrix} -9 & 9 & 18 \\ -1 & -1 & -6 \\ -2 & -2 & 6 \end{bmatrix}$

9 Exercise 9

- a. and b. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$
- c. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- d. This matrix cannot be inverted; it's determinant is 0. Also: we lose information when applying the matrix; this can not be generated out of thin air.

10 Exercise 10

If we simply rotate a vector, then the transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

satisfies $x'^2 + y'^2 = x^2 + y^2$. This means that the vector is rotated along a circle. However, if we want a rotate a vector along an ellipse of width a and height b, then the transformation must satisfy $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$. One can express this condition as

$$\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}^{-1} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\frac{a}{b}\sin\phi \\ \frac{b}{a}\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

So the solution is $\begin{bmatrix} \cos\phi & -\frac{1}{4}\sin\phi \\ 4\sin\phi & \cos\phi \end{bmatrix}$. See the last page of maths lecture (scan) 4.

11 Exercise 11

We'll do it the way we did a related example in maths lecture 4.

First we take a vector along the line $2x - y = 0$. Such a vector is simply $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Let the matrix elements of transformation matrix A be denoted as $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then A operating on this vector will keep the vector unchanged, which means $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

We then take a vector perpenticular to the line $2x - y = 0$. Such a vector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. The matrix A operating on this vector will flip its sign, i.e., $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

These two matrix equations give us four equations, to be solved for the four unknowns a_{11}, a_{12}, a_{21} and a_{22}. The solution is $a_{11} = -3/5, a_{12} = 4/5, a_{21} = 4/5$ and $a_{22} = 3/5$. So the matrix A is $\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$.

12 Exercise 12

- a. $A = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
- b. Work it out yourself.
- c. $AA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Convince yourself that it is a rotation by $90°$.