Tutorial 5: The Matrix strikes back

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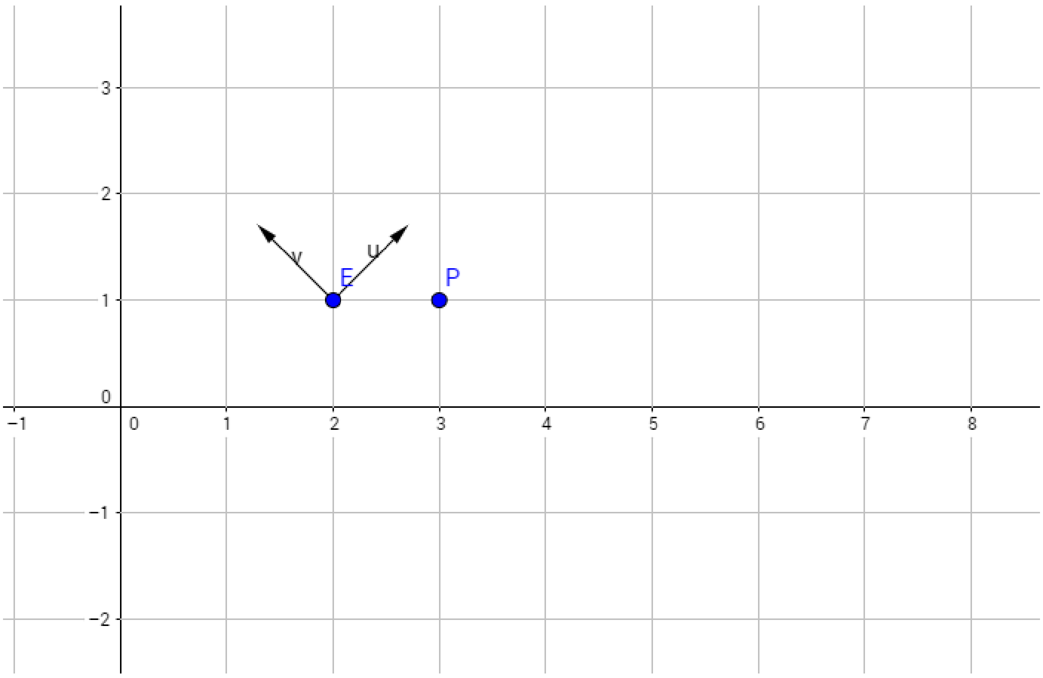
June 19, 2017

Exercise 1

a



b

Remember, $\vec{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we want to write P_{xy} as a linear combination of $\vec{b}_1$ and $\vec{b}_2$. So we get the general form of $P_{xy} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Next we want to write P_{uv} as a linear combination of $\vec{u} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$. The general form of $P_{uv} = u \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} + v \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$

c

We need a matrix so that $P_{xy} = M_1 \cdot P_{uv}$. We want to rotate so that $\vec{u}$ is on the x-axis, we rotate with $\phi = \frac{\pi}{4}$ We do this with matrix:

$$\frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we translate the camera to point $E = (2, 1)$.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 2 \\ \sqrt{2} & \sqrt{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we need a matrix so that $P_{uv} = M_2 \cdot P_{xy}$. First we translate the camera to the origin.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then we rotate it with $\phi = -\frac{\pi}{4}$,

$$\frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This gives us:

$$M_2 = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & -3\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

d

Now fill in $P_{xy} = (3, 1)$ into M_2 to calculate the camera coordinates.

$$P_{uv} = M_2 \cdot P_{xy} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & -3\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

So $P_{uv} = (\sqrt{2}, -\sqrt{2})$.

Exercise 2

a

The $\vec{up}$ vector needs to be perpendicular to the view vector, and it needs to point upwards. Otherwise we would get a tilted camera.

We can calculate this by getting a perpendicular vector on the view vector and the up vector.

$$\vec{side} = \vec{view} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and then to get the up vector.

$$\vec{up} = \vec{view} \times \vec{side}$$

b

$$\vec{side} = \vec{view} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 3

a

The orthographic view volume is defined as $[[l, r], [t, b], [n, f]]$. So $x \in [l, r]$, $y \in [t, b]$ and $z \in [n, f]$.

b

The canonical view volume is defined as $[[-1, 1], [-1, 1], [-1, 1]]$. So $x \in [-1, 1]$, $y \in [-1, 1]$ and $z \in [-1, 1]$.

c

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{l+r}{l-r} \\ 0 & \frac{2}{t-b} & 0 & \frac{b+t}{b-t} \\ 0 & 0 & \frac{-2}{n-f} & \frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We show that $[l,r] \to [-1,1]$, $[b,t] \to [-1,1]$ and $[n,f] \to [-1,1]$.

$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{l+r}{l-r} \\ 0 & \frac{2}{t-b} & 0 & \frac{b+t}{b-t} \\ 0 & 0 & \frac{-2}{n-f} & \frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l & r \\ b & t \\ n & f \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{l+r}{l-r} - \frac{2l}{l-r} & \frac{l+r}{l-r} - \frac{2r}{l-r} \\ \frac{b+t}{b-t} - \frac{2b}{b-t} & \frac{b+t}{b-t} - \frac{2t}{b-t} \\ \frac{n+f}{n-f} - \frac{2n}{n-f} & \frac{n+f}{n-f} - \frac{2f}{n-f} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Exercise 4

a

A point needs to have depth, and depth translates to homogeneous coordinates.

b

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \vec{v} & \vec{v}_h \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x & hx \\ y & hy \\ z & hz \\ 1 & h \end{bmatrix} =$$

$$\begin{bmatrix} d_1 + c_1 z + b_1 y + a_1 x & hd_1 + c_1 hz + b_1 hy + a_1 hx \\ d_2 + c_2 z + b_2 y + a_1 x & hd_2 + c_2 hz + b_2 hy + a_2 hx \\ d_3 + c_3 z + b_3 y + a_1 x & hd_3 + c_3 hz + b_3 hy + a_3 hx \\ 1 & h \end{bmatrix} =$$

$$\begin{bmatrix} d_1 + c_1 z + b_1 y + a_1 x & h(d_1 + c_1 z + b_1 y + a_1 x) \\ d_2 + c_2 z + b_2 y + a_1 x & h(d_2 + c_2 z + b_2 y + a_2 x) \\ d_3 + c_3 z + b_3 y + a_1 x & h(d_3 + c_3 z + b_3 y + a_3 x) \\ 1 & h \end{bmatrix} = \begin{bmatrix} \vec{t} & \vec{t}_h \end{bmatrix}$$

$$\text{After homogenization } \vec{t}_h = \begin{pmatrix} d_1 + c_1 z + b_1 y + a_1 x \\ d_2 + c_2 z + b_2 y + a_1 x \\ d_3 + c_3 z + b_3 y + a_1 x \\ 1 \end{pmatrix} \text{ which is equal } \vec{t}.$$

c

Our extended framework is:

$$M = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & fn \\ 0 & 0 & -1 & n+f \end{bmatrix}$$

Assume we have two vectors $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ a \\ 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ b \\ 1 \end{pmatrix}$, where $a \leq b$.

multiplying both of these vectors with M we get: $\vec{a_m} = \begin{pmatrix} 0 \\ 0 \\ fn \\ n+f-a \end{pmatrix}$

and $\vec{b_m} = \begin{pmatrix} 0 \\ 0 \\ fn \\ n+f-b \end{pmatrix}$. After homogenization these become:

$$\vec{a}_{hm} = \begin{pmatrix} 0 \\ 0 \\ \frac{fn}{n+f-a} \\ 1 \end{pmatrix} \text{ and } \vec{b}_{hm} = \begin{pmatrix} 0 \\ 0 \\ \frac{fn}{n+f-b} \\ 1 \end{pmatrix}$$

Now we need to check if the order of z is preserved.

$$\frac{fn}{n+f-a} \leq \frac{fn}{n+f-b} \Leftrightarrow \frac{1}{n+f-a} \leq \frac{1}{n+f-b}$$
$$\Leftrightarrow n+f-b \leq n+f-a \Leftrightarrow -b \leq -a \Leftrightarrow b \geq a$$

The order of z is preserved.

Exercise 5

The view vector is the vector specifying the looking direction. In this case it is V = (3, 4, 12)

Exercise 6

First, we need to construct an orthonormal basis $(\vec{u}, \vec{v}, \vec{w})$ for the camera. Here, $\vec{w}$ is simply the normalized opposite view vector (note that we look into the negative $\vec{w}$ direction), so $\vec{w} = -\vec{v}/||\vec{v}||$. Filling up the numbers gives $\vec{w} = (3/13, 4/13, 12/13)$ The vector $\vec{u}$ is perpendicular to the plane spanned by $\vec{w}$ and the up vector. So we take the cross product of those two vectors, and normalize: $\vec{u} = \vec{up} \times \vec{w}/(||\vec{up} \times \vec{w}||)$. In our concerte case we get $\vec{u} = (4/\sqrt{17}, 0, -1/\sqrt{17})$.

Finally, $\vec{v}$ is perpendicular to $\vec{w}$ and $\vec{u}$, so $\vec{v} = \vec{w} \times \vec{u}$. In our case this gives $\vec{v} = (-4/13\sqrt{17}, 51/13\sqrt{17}, -16/13\sqrt{17})$.

We find that the matrix M_{cam} by first translating over $(-x_e, -y_e, -z_e)$, and next multiplying by the matrix where the rows are formed by $\vec{u}, \vec{v}$, and $\vec{w}$ (completed with zeros and ones in the appropriate places). The latter matrix aligns the camera coordinate system with the global world coordinate system. If you work out the numbers, you should end up wit the matrix:

$$\begin{pmatrix} 4/\sqrt{17} & 0 & -1/\sqrt{17} & -1/\sqrt{17} \\ -4/(13\sqrt{17}) & 51/(13\sqrt{17}) & -16/(13\sqrt{17}) & -500/(13\sqrt{17}) \\ 3/13 & 4/13 & 12/13 & -470/13 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 7

This matrix can be found in the lecture slides, and in the textbook on page 152. Filling in the numbers is left as an exercise for the reader.

Exercise 8

See exercise 3. You can also find it in the slides, and in the textbook on page 145. Completing the answer is a matter of filling in the numbers and doing the matrix multiplications. Notice however, that you are strongly advised to not just copy the formulas and fill in the numbers, but to make sure you understand how we got to this matrix in the first place!

Exercise 9

Anyone brave enough to compute all matrix multiplications such that the resulting matrix maps the point (7,16,18) onto the center of the image deserves the utmost respect. I didn't do it myself; the result is not very important, but understanding the whole procedure is.