Q6. We take the three points $A = (1,1,1)$, $B = (4,3,4)$, $C = (2,4,3)$.

Vector $\overline{AB} = (3,2,3) = (4,3,4) - (1,1,1)$

Vector $\overline{AC} = (1,3,2) = (2,4,3) - (1,1,1)$

$\overline{AB} \times \overline{AC} = (-5,-3,7)$ is a vector perpendicular to the plane ABC.

Eqn. of the plane is therefore $-5x - 3y + 7z + p = 0$

p is determined by letting $(1,1,1)$ lie on the plane,

i.e., $p = 5+3-7 = 1$.

Eqn. of the plane is $\boxed{5x + 3y - 7z - 1 = 0}$

Q7. $\boxed{0}$ Since the first and the third columns are parallel.

Q8. (a) $MM = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}$

$MM = I \Rightarrow a^2 = 1;$ $\boxed{a = \pm 1}$

(b) To describe rotation

$N^T N = N N^T = I$

$\begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} c^2 & 0 \\ 0 & b^2 \end{bmatrix} = I$

i.e., $c^2 = b^2 = 1. \Rightarrow c = \pm 1, \; b = \pm 1.$

But N then also needs to satisfy the form

$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \Rightarrow \cos\phi = 0 \Rightarrow \sin\phi = \pm 1$

i.e., $b = -c \Rightarrow \boxed{(b,c) = \pm(1,-1)}$

Q9. (a) Note that the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is parallel to the line L, so it remains unchanged upon reflection; i.e., $\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow[\text{reflection}]{} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Think of the vector as an arrow and the line as a mirror

(b) On the other hand $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is perpendicular to L.
i.e, upon reflection it changes sign.

$$\boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \xrightarrow[\text{reflection}]{} -\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

(c) $2x-y+3=0$ and $8x-4y+5=0$ are parallel.
This question is identical to Q11 of tutorial 4.
So, we can use the ingredients of (a) and (b) above.
Assume that the matrix elements are $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} a_{11} + 2a_{12} = 1 \rightarrow e1 \\ a_{21} + 2a_{22} = 2 \rightarrow e2 \end{cases}$$

and $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} 2a_{11} - a_{12} = -2 \rightarrow e3 \\ 2a_{21} - a_{22} = 1 \rightarrow e4 \end{cases}$$

$2 \times e1 - e3 \Rightarrow 5a_{12} = 4 \Rightarrow a_{12} = 4/5$

$e1 + 2 \times e3 \Rightarrow 5a_{11} = -3 \Rightarrow a_{11} = -3/5$

$e2 + 2 \times e4 \Rightarrow 5a_{21} = 4 \Rightarrow a_{21} = 4/5$

$2 \times e2 - e4 \Rightarrow 5a_{22} = 3 \Rightarrow a_{22} = 3/5$

This question is about reflecting vectors, not points; one cannot reflect points with a 2x2 matrix!

$$\boxed{\text{Answer} \rightarrow \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}}$$

Q10. The surface normal vector at point P is

$$\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

The tangent plane has the form $2x+y+2z+\phi=0$
The fact that this plane passes through P means

$$\phi = -(2 \times 5 + 5 + 2 \times 2) = -19$$

$$\boxed{\text{Answer} \rightarrow 2x+y+2z-19=0}$$

Q11. We need to combine two operations:
reflections about the y-axis $\rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

followed by scaling the y-axis by a factor 2
$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. The resulting matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}$$

You can switch the order of the operations and get the same answer. $\boxed{\text{Inverse is } \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}}$.

Q12. (a) $x \rightarrow x' = 3x$ The centre $(x_0, y_0) = (2, 3)$
$y \rightarrow y' = 2y$ gets mapped to $(x_0', y_0') = (6, 6)$

$$(x - x_0)^2 + (y - y_0)^2 = 1$$

$$\Rightarrow \left(\frac{x'}{3} - \frac{x_0'}{3}\right)^2 + \left(\frac{y'}{2} - \frac{y_0'}{2}\right)^2 = 1$$

$$\Rightarrow \frac{(x' - x_0')^2}{9} + \frac{(y' - y_0')^2}{4} = 1$$

i.e., $\boxed{\frac{(x'-6)^2}{9} + \frac{(y'-6)^2}{4} = 1}$

(b) This is $\boxed{\text{an ellipse}}$.

Q13. (a) $\boxed{M = M_r M_t}$

(b) $\boxed{M_t = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$ $\boxed{M_r = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$

$$\boxed{M = \begin{bmatrix} \cos\phi & 0 & \sin\phi & x_0\cos\phi + z_0 \sin\phi \\ 0 & 1 & 0 & y_0 \\ -\sin\phi & 0 & \cos\phi & z_0\cos\phi - x_0 \sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$