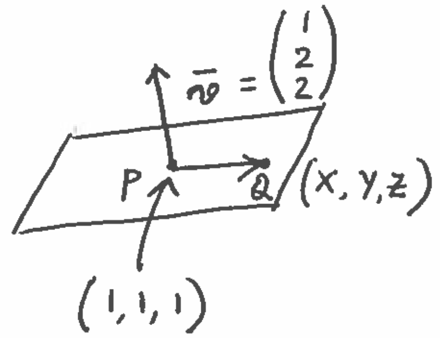
Q1.



Two ways to solve this.

a) We assume that the implicit form eq. of the plane is $Ax + By + Cz + D = 0$ where $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ denotes a vector perpendicular to the plane.

So we choose $A = 1$, $B = 2$, $C = 2$, then all we need is to determine D, which can be obtained from the fact that $P = (1,1,1)$ is a point on the plane. I.e., $1\times1 + 2\times1 + 2\times1 + D = 0 \Rightarrow D = -5$

So the eqn. of the plane is

$$\boxed{x + 2y + 2z - 5 = 0}$$

b) Alternatively consider a point $Q = (x, y, z)$ on the plane. The vectors $\overline{PQ} = \begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ are perpendicular to each other. I.e., $\overline{PQ} \cdot \bar{v} = 0$

$$\Rightarrow (x-1)1 + (y-1)2 + (z-1)2 = 0$$

$$\Rightarrow \boxed{x + 2y + 2z - 5 = 0}$$

Q4. The (parametric) eqn. of a line starting from $(8,4,0)$ along the vector $\bar{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The points of intersection satisfies the eqn.

$$(x-3)^2 + (y-4)^2 + z^2 = 25$$

$$\Rightarrow (t+5)^2 + t^2 = 25$$

$$\Rightarrow t(t+5) = 0 \Rightarrow t = 0 \text{ or } -5.$$

I.e., the intersection points are $(8,4,0)$ and $(8-5, 4, -5)$

<u>Point P1</u> $\nearrow$ $(8,4,0)$

<u>Point P2</u> $\rightarrow (3, 4, -5)$

The centre of the sphere is located at $(3, 4, 0)$, so by construction the vectors

Point O

$\overline{OP_1}$ and $\overline{OP_2}$ are the outward normals.

$$\begin{pmatrix} 8-3 \\ 4-4 \\ 0-0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 3-3 \\ 4-4 \\ -5-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$$

So the outward unit normal vectors are $\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}$

Q6. $\bar{v} \times \bar{w}$ is perpendicular to $\bar{v}$

$$\Rightarrow \boxed{\bar{v} \cdot (\bar{v} \times \bar{w}) = 0}$$

Q10. Although this problem looks like a problem in 3 dimensions, note that all points (A, B, P) have zero y-coordinates. So, the triangles $EA'B'$ and EAB are confined to the x-z plane. We use this information to simplify the solution, as we confine the solution below fully to the x-z plane.

The unit vector along $\overline{AB}$ $= \dfrac{1}{\sqrt{(20-8)^2 + (12-3)^2}} \begin{pmatrix} 20-8 \\ 12-3 \end{pmatrix}$

$$= \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

So, the location of P is $\left(8 + \frac{4l}{5}, 3 + \frac{3l}{5}\right)$ on the x-z plane.

The slope-intercept form equations for EA and EP are

$$z = \frac{7}{8} x - 4 \quad \text{and}$$

$$z = \frac{7 + \frac{3l}{5}}{8 + \frac{4l}{5}} x - 4 \quad \text{respectively.}$$

These lines intersect the screen (xy-plane), for which $z = 0$, at

$$x = \frac{32}{7} \text{ (at } A'\text{) and } \frac{32 + 16l/5}{7 + 3l/5} \text{ (at } P'\text{) resp.}$$

$$\text{So,}\quad t = \frac{32+16l/5}{7+3l/5} - \frac{32}{7}$$

$$= \frac{112l/5 - 96l/5}{7\left(7+\frac{3l}{5}\right)} = \boxed{\frac{16l}{7(3l+35)}}$$

Q11. We have 3 points A, B, C on the plane; take any two vectors, eg. $\overline{AB} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ and $\overline{AC} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$

The cross-product of these two vectors is perpendicular to the triangular plane. The cross product is

$$\bar{v} = \overline{AB} \times \overline{AC} = \cancel{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ -1 \\ -6 \end{pmatrix} \text{ and } \hat{v} = \cancel{\begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}} \frac{1}{\sqrt{41}}\begin{pmatrix} 2 \\ -1 \\ -6 \end{pmatrix}$$

So the answer is $\boxed{\cancel{\pm \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}}}$ $\boxed{\pm \frac{1}{\sqrt{41}}\begin{pmatrix} 2 \\ -1 \\ -6 \end{pmatrix}}$

Note: a surface has only two normal vectors differing in sign, so the answer is independent of whether you choose $\overline{BC}$ & $\overline{AC}$ or $\overline{AB}$ & $\overline{BC}$ pairs to work with in terms of cross-products.

Q12. This problem is in the same vein as Q10.

The unit vector along $\overline{AB} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$.

location of $P = \left(2+\frac{3l}{5},\ 4+\frac{4l}{5}\right)$

Slope-intercept forms of the straight lines EA and EP are

$$y = 4x-4 \quad \text{and} \quad y = \frac{8+4l/5}{2+3l/5}x - 4 \quad \text{respectively}$$

These lines intersect the x-axis at

$\underline{x=1}$ and at $x = \dfrac{8+12l/5}{8+4l/5} = \underline{\dfrac{2+3l/5}{2+l/5}}$

Point A′ ... Point P′

respectively.

$$\text{So,}\quad t = \frac{2+3l/5}{2+l/5} - 1 = \boxed{\frac{2l}{l+10}}.$$

Q14. The vector $\bar{AB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. A vector perpendicular to that is $\bar{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, from which we know that the implicit form looks like

$$-x + y + c = 0$$

Since the line passes through $(1, 2)$,

$$c = -1$$

i.e, the equation is $\boxed{x - y + 1 = 0}$

We let $\bar{v}$ pass through $P = (1, 5)$

The parametric eqn. of the line passing through P along $\bar{v}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and it intersects the line $x - y + 1 = 0$ at S, for which

$$(1-t) - (5+t) + 1 = 0 \Rightarrow t = -\frac{3}{2}$$

I.e., the co-ordinates of S are $\left(1 + \frac{3}{2}, 5 - \frac{3}{2}\right) = \boxed{\left(\frac{5}{2}, \frac{7}{2}\right)}$