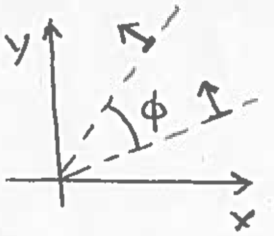
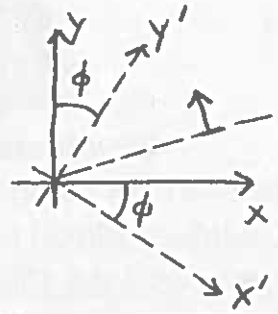
Rotation matrices



$$M = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$



$$\left.\begin{aligned} \hat{x}' &= \hat{x}\cos\phi - \hat{y}\sin\phi \\ \hat{y}' &= \hat{x}\sin\phi + \hat{y}\cos\phi \end{aligned}\right\} \rightarrow M$$

$\boxed{\text{Row elements of } M \text{ are the components of the new axes unit vectors in the old co-ordinate system}}$

Position in new co-ordinate = M position in old coordinate

" " old " = M^{-1} " " new "

$$M = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix} \qquad \begin{aligned} X_x^2 + X_y^2 &= 1 \\ Y_x^2 + Y_y^2 &= 1 \end{aligned}$$

$$M^T M = M M^T = I \Rightarrow M^{-1} = M^T$$

These matrices are called <u>orthonormal</u> matrices.

In 3D

$$M = \begin{pmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{pmatrix}$$

<u>Change of handedness</u> 2D: $y \rightarrow -y$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad M^T = M^{-1}$$

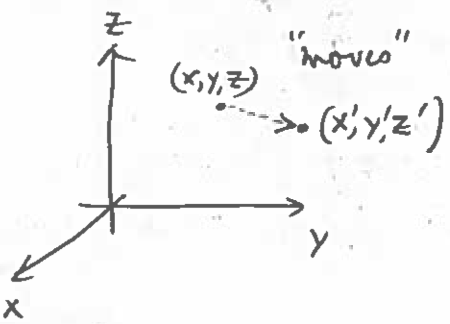
3D: $z \rightarrow -z$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad M^T = M^{-1}$$

Transformations

In the last class

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



I want to make another "move" from $(x', y', z') \longrightarrow (x'', y'', z'')$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = B \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = BA \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

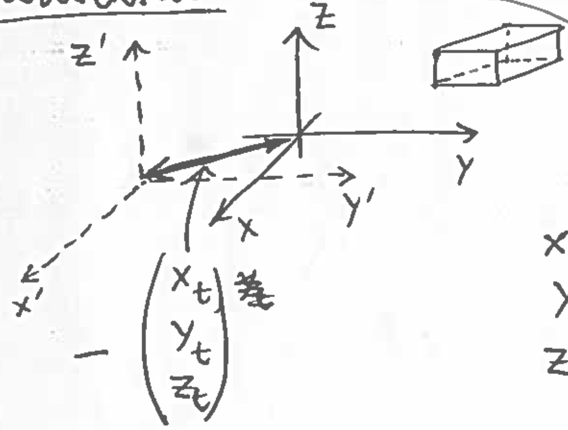
Caution: BA <u>NOT</u> AB, since AB means $(x,y,z) \rightarrow (x_1, y_1, z_1) \rightarrow (x_2, y_2, z_2)$

and $(x'', y'', z'') \neq (x_2, y_2, z_2)$

goes back to $AB \neq BA$

Such a "move" is called <u>transformation</u>. Transformation comes about if we change our measuring stick like meter to feet.

Translation



Here, we move the origin of the co-ordinate system to a new position.

$$x' = x + x_t$$
$$y' = y + y_t$$
$$z' = z + z_t$$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

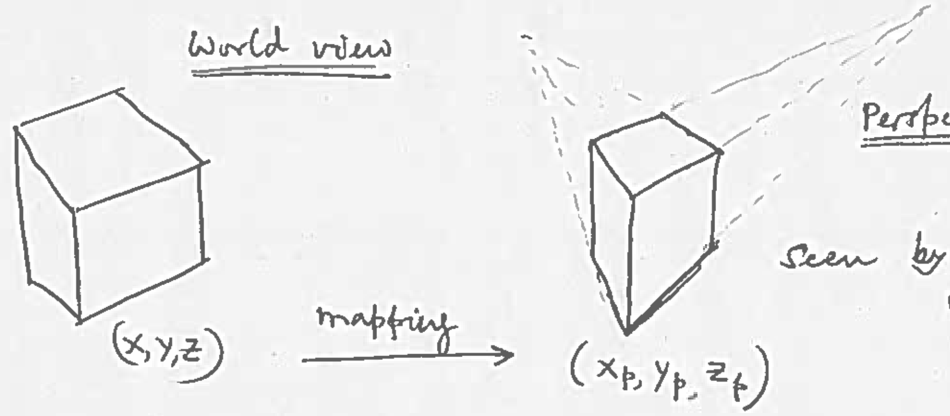
but we want to express it as $\bar{V}' = M\bar{V}$

Enlarge our ~~vect~~ 3-d vector by a fictitious dimension

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

From now on we'll work in terms of these "enlarged" vectors.

Goal for ~~projection~~ transformations — why do we need them?



Seen by a camera or an eye

Then in the end, you need to project ~~the~~ what the camera sees on to a screen.

The viewport transformation

Mapping of $(x,y,z) \in [-1,1]^3$ to pixels. $n_x \times n_y$

$$n_x \to [0, n_x] \qquad n_y \to [0, n_y]$$

$$x_s = \frac{n_x}{2} x_c + \frac{n_x}{2}$$

$$y_s = \frac{n_y}{2} y_c + \frac{n_y}{2}$$

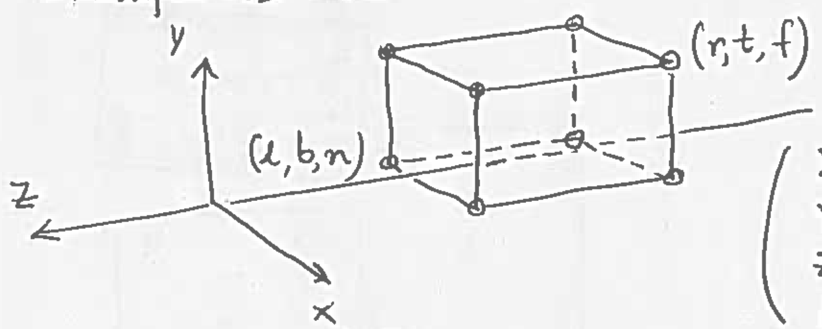
$$z_s = z_c$$

$$\begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M_{vp}} \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix}$$

the orthographic transformation

viewing direction is perpendicular to the image plane for all points on the object. is called orthographic view.

Example:



$$\begin{pmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{pmatrix} = M_{ortho} \begin{pmatrix} X_m \\ Y_m \\ Z_m \\ 1 \end{pmatrix}$$

$$M_{ortho} = \underbrace{\begin{pmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{scale}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & -\frac{l+r}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{move}} = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{l+r}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The camera transformation

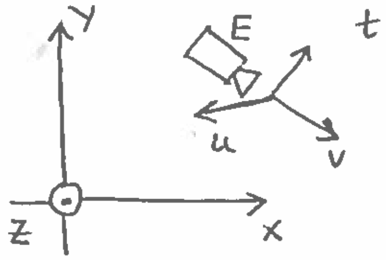
In the world space

- camera positioned at E
- Viewing direction $\hat{v}$

So to go from $\begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$ to $\begin{pmatrix} x_{cam} \\ y_{cam} \\ z_{cam} \\ 1 \end{pmatrix}$

first move to (x_E, y_E, z_E) $\longrightarrow$ $\underbrace{\begin{pmatrix} 1 & 0 & 0 & -x_E \\ 0 & 1 & 0 & -y_E \\ 0 & 0 & 1 & -z_E \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{move}}$

Then rotate your axis



$\underbrace{\begin{pmatrix} x_t & y_t & z_t & 0 \\ x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{rotate}}$

World to camera

$$\begin{pmatrix} x_m \\ y_m \\ z_m \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} x_t & y_t & z_t & 0 \\ x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_E \\ 0 & 1 & 0 & -y_E \\ 0 & 0 & 1 & -z_E \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M_c} \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$$

World $\longrightarrow$ camera $\longrightarrow$ (orthographic) canonical $\longrightarrow$ screen

M_m $\qquad$ M_{ortho} $\qquad$ M_{vp}

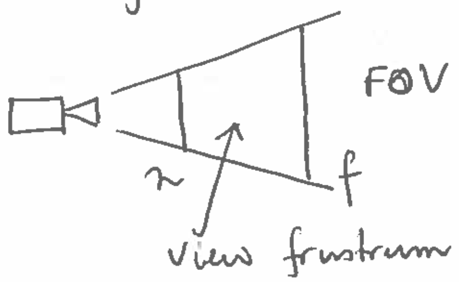
$$\begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix} = M_{vp} M_o M_c \begin{pmatrix} x_w \\ y_w \\ z_w \\ 1 \end{pmatrix}$$

Up until now

- We have used the camera information, but only in an incomplete manner.

For example, we have ignored that in ordered to be seen by a camera, all light rays emanating from an object has to pass through a lens. Hence a camera

- has a field of view
- an object that is nearer appears bigger etc.



view frustrum

defined by the orthographic volume

- near and far planes n and f

$$\boxed{\begin{array}{llll} - \text{ left} & \text{"} \text{ right} & \text{"} \; l & \text{"} \; r \\ - \text{ top} & \text{"} \text{ bottom} & \text{"} \; t & \text{"} \; b. \end{array}}$$

mention later!

But to deal with it we need projective transformation.

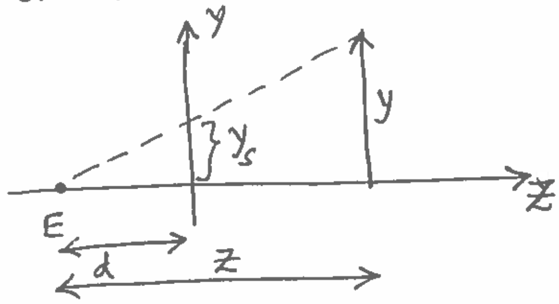
We again need an extra dimension like translation.

For translation, the extra dimension was always 1 and the last row for the transformation matrix was $(0\ 0\ 0\ 1)$. We'll now relax that such that

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{\omega} \end{bmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a & b & c & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Then $(x', y', z') = (\tilde{x}/\tilde{\omega}, \tilde{y}/\tilde{\omega}, \tilde{z}/\tilde{\omega})$

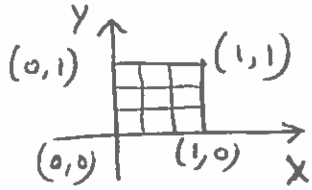
Lets use it in 2D:



$$y_s = \frac{yd}{z}.$$

Another example:

$$M = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

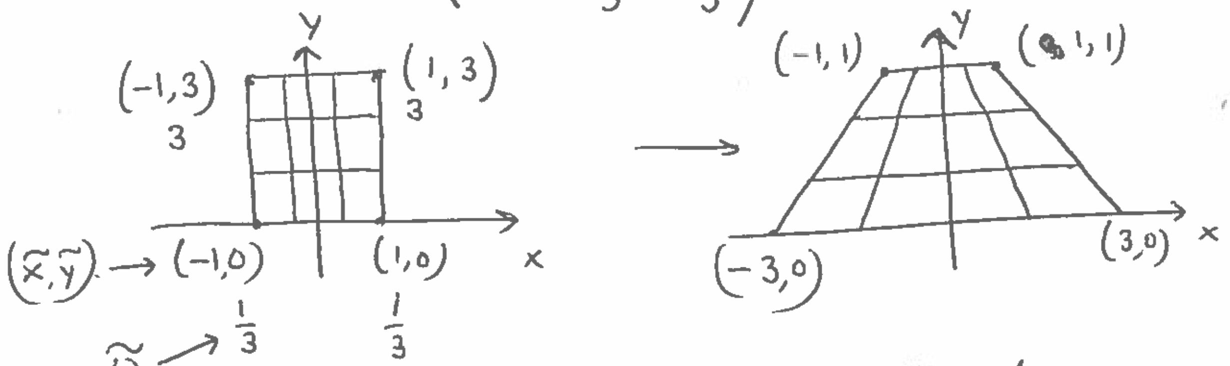


$$(0,0,1) \longrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1/3 \end{pmatrix}$$

$$(0,1,1) \longrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$(1,0,1) \longrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

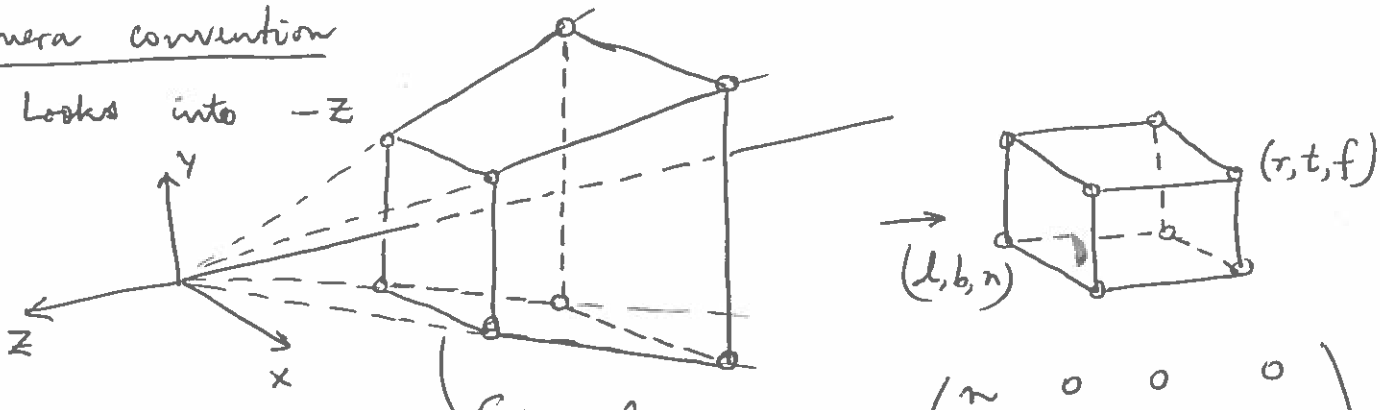
$$(1,1,1) \longrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$



We are now ready to deal with the camera!

Camera convention

Looks into $-z$



$$y_s = \frac{n}{z}\, y$$

$$x_s = \frac{n}{z}\, s$$

$$P = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

No unique solution. One specific matrix is

$$P\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{pmatrix}$$

$$\cancel{z = n + (f-n)q \qquad q \in [0,1]}$$

$$\cancel{-nf + (n+f)z = n(n+f) + (f^2-n^2)q - nf = n^2 + (f^2-n^2)q}$$

$$\cancel{\frac{z(n+f) - nf}{z} = n+f-}$$

Distortion in the transformed z-coordinates however

$$\text{Preserves}\begin{cases} z = n \longrightarrow z' = n \\ z = f \longrightarrow z' = f \\ z_1 > z_2 \longrightarrow z_1' > z_2' \end{cases}$$

$$P^{-1} = \begin{pmatrix} \frac{1}{n} & 0 & 0 & 0 \\ 0 & \frac{1}{n} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{fn} & \frac{n+f}{fn} \end{pmatrix}$$

or also $\begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & fn \\ 0 & 0 & -1 & n+f \end{pmatrix}$

$$M_p = M_o P$$

$$M = M_{vp} M_o P M_c$$