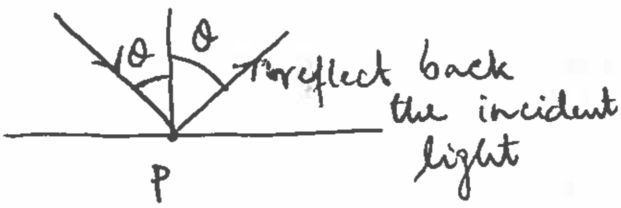
Specular reflection

idealised smooth surface

"glossy" photograph

non-smooth surface (at a very fine scale)

matt photograph



glossy appearance

matt appearance

How we do this will be discussed in the graphics lectures... but today we'll learn about the mathematics behind that.

- Averages
- Weighted average
- probability (density)
- measure
- expectation
- law of large numbers
- Sampling of various sorts
- Monte Carlo

$\Big\}$ Ch. 14.

<u>Averages</u> Avg. Test score school 1   Average test score school 2

20   30

Avg. test score $\frac{20+30}{2} = 25.$

<u>weighted average</u> school 1 has 20 students   school 2 has 80 students

Avg. test score $\frac{20\times 20 + 30\times 80}{20+80} = 28$

$$\bar{S} = 20 \times \frac{20}{20+80} + 30 \times \frac{80}{20+80}$$

$$S_1 \times \underbrace{\frac{N_1}{N_1+N_2}}_{p_1} + S_2 \times \underbrace{\frac{N_2}{N_1+N_2}}_{p_2}$$

In general $\bar{s} = s_1 p_1 + s_2 p_2 + s_3 p_3 + \cdots + s_N p_N$

$p_1 + p_2 + p_3 + \cdots + p_N = 1$ definition.

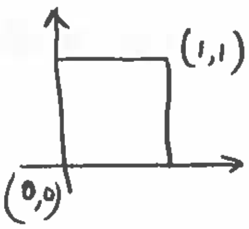
Domain or Space of schools : $1 \ldots N$: Discrete space

"probability" $\equiv$ proportion of students in this space : $p_1, \cdots, p_N$

"random" variable : $s_1, \cdots s_N$

Take another example:

I have a unit square in 2D : $[0,1]^2 \in \mathbb{R}^2$



Any point on this square is denoted by (x,y)

Space is continuous. Each point (x,y) radiates an Intensity

$I(x,y) = xy$ of wavelength $\lambda(x,y)$.

Total radiation $I_{tot} = \int I(x,y)\, dx\, dy$

$$= \int x y \, dx \, dy = \frac{1}{4}$$

Probabity of a ray reaching the eye from $[(x,y),(x+dx, y+dy)]$ is $\dfrac{I(x,y)\,dx\,dy}{\frac{1}{4}} = \underbrace{4\, I(x,y)}_{\text{probability density}} dx dy$

$$\bar{\lambda} = \int \underbrace{\lambda(x,y)}_{\text{variable}} \underbrace{4\, I(x,y)}_{\substack{P(x,y) \\ \text{prob. density}}} \underbrace{dx\, dy}_{\text{measure}}$$

Measure theory in Mathematics, but in computer graphics it is often the area of a surface.

Cartesian : (x,y), $dA = dx\,dy$
Circular : (r,ϕ); $dA = r\,dr\,d\phi$
Spherical : (θ,ϕ); $dA = r^2 \sin\theta\, d\theta\, d\phi$

Why not $dr\,d\phi$? or $d\theta\,d\phi$?

There are relations among these co-ordinate systems.

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$dx = dr\cos\phi - r\sin\phi\,d\phi$$
$$dy = dr\sin\phi + r\cos\phi\,d\phi$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}}_{A} \begin{pmatrix} dr \\ d\phi \end{pmatrix}$$

$$J = |\det A| = r$$
$$dx\,dy = (\det A)\,dr\,d\phi$$

Jacobian decides that the total number of points in $dx\,dy$ is the same as $r\,dr\,d\phi$

$\bar{\lambda}$ = <u>expectation value</u>

$$= E[\lambda(x,y)] = \int_S \underset{\uparrow}{\lambda(s)}\ \underset{\uparrow}{P(s)}\ d\mu(s)$$

domain, variable, probability density, measure

Monte Carlo!

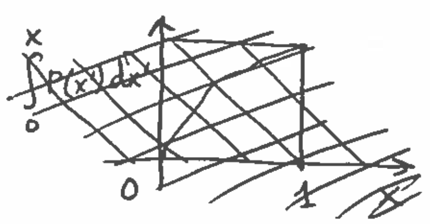
So to calculate $E[\lambda(s)]$, on a computer we need to choose samples according to the probability distribution $P(s)$ and define an estimate

$$E_N[\lambda(s)] = \frac{1}{N}\sum_i \lambda(s_i)$$

First question is; how do we choose a point in S that satisfies $P(s)$?

1D: Uniform probability in an interval $[0,1]$

bounded domain



for a given x

$$P(x)\,dx = \begin{cases} dx & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

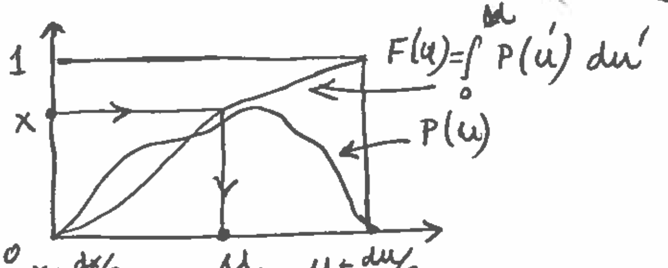
$$\int_0^1 P(x)\,dx = 1$$

~~A function of x : $u(x)$~~

~~$|P(u)\,du| = |P(x)\,dx|$~~

~~$P(u) = P(x) \Big/ \left|\frac{du}{dx}\right|$~~

~~Using $P(x) = 1$; $P(u) = \frac{1}{\left|\frac{du}{dx}\right|}$~~



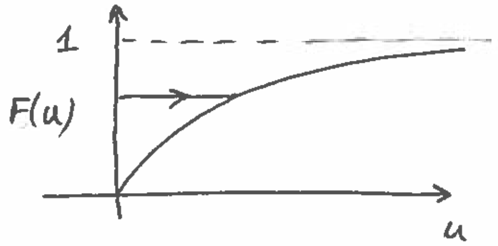
$$\int_{x-\frac{dx}{2}}^{x+dx/2} P(x)\,dx = \int_{u-\frac{du}{2}}^{u+du/2} P(u)\,du$$

Drawing exponentially distributed variables

$$P(u) = \lambda \; e^{-\lambda u}$$

$$\int_0^\infty P(u)\,du = \lambda \int_0^\infty e^{-\lambda u}\,du = -e^{-\lambda u}\Big|_0^\infty = 1 .$$

$$F(u) = \int_0^u P(u')\,du' = -e^{-\lambda u'}\Big|_0^u = 1 - e^{-\lambda u}$$



$$F(u) = x$$

$$e^{-\lambda u} = 1 - x$$

$$u = \frac{-1}{\lambda} \ln(1-x)$$

This is not the only way to do it.

Drawing Gaussian distributed variable

x_1, x_2 ; $\quad u_1 = \sqrt{-2\ln x_1}\;\cos(2\pi x_2)$

$$u_2 = \sqrt{-2\ln x_1}\;\sin(2\pi x_2)$$

$$P(u_1) = \frac{1}{\sqrt{2\pi}} e^{-u_1^2/2}$$; similarly $P(u_2)$

7.2 Transformation Method: Exponential and Normal Deviates

In the previous section, we learned how to generate random deviates with a uniform probability distribution, so that the probability of generating a number between x and $x + dx$, denoted $p(x)dx$, is given by

$$p(x)dx = \begin{cases} dx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.2.1}$$

The probability distribution $p(x)$ is of course normalized, so that

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{7.2.2}$$

Now suppose that we generate a uniform deviate x and then take some prescribed function of it, $y(x)$. The probability distribution of y, denoted $p(y)dy$, is determined by the fundamental transformation law of probabilities, which is simply

$$|p(y)dy| = |p(x)dx| \tag{7.2.3}$$

or

$$p(y) = p(x)\left|\frac{dx}{dy}\right| \tag{7.2.4}$$

Exponential Deviates

As an example, suppose that $y(x) \equiv -\ln(x)$, and that $p(x)$ is as given by equation (7.2.1) for a uniform deviate. Then

$$p(y)dy = \left|\frac{dx}{dy}\right| dy = e^{-y}dy \tag{7.2.5}$$

which is distributed exponentially. This exponential distribution occurs frequently in real problems, usually as the distribution of waiting times between independent Poisson-random events, for example the radioactive decay of nuclei. You can also easily see (from 7.2.4) that the quantity y/λ has the probability distribution $\lambda e^{-\lambda y}$.

So we have

```
#include <math.h>

float expdev(long *idum)
Returns an exponentially distributed, positive, random deviate of unit mean, using
ran1(idum) as the source of uniform deviates.
{
    float ran1(long *idum);
    float dum;

    do
        dum=ran1(idum);
    while (dum == 0.0);
    return -log(dum);
}
```

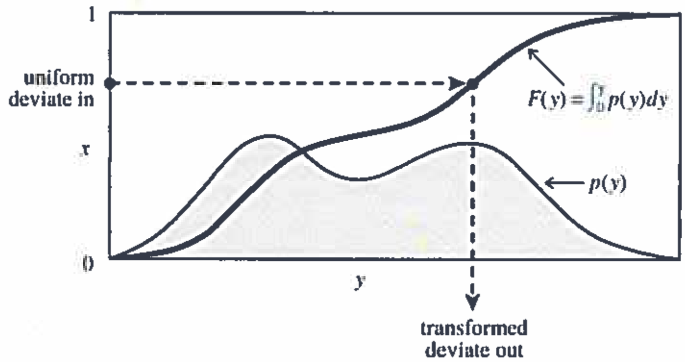


Figure 7.2.1. Transformation method for generating a random deviate y from a known probability distribution $p(y)$. The indefinite integral of $p(y)$ must be known and invertible. A uniform deviate x is chosen between 0 and 1. Its corresponding y on the definite-integral curve is the desired deviate.

Let's see what is involved in using the above *transformation method* to generate some arbitrary desired distribution of y's, say one with $p(y) = f(y)$ for some positive function f whose integral is 1. (See Figure 7.2.1.) According to (7.2.4), we need to solve the differential equation

$$\frac{dx}{dy} = f(y) \tag{7.2.6}$$

But the solution of this is just $x = F(y)$, where $F(y)$ is the indefinite integral of $f(y)$. The desired transformation which takes a uniform deviate into one distributed as $f(y)$ is therefore

$$y(x) = F^{-1}(x) \tag{7.2.7}$$

where F^{-1} is the inverse function to F. Whether (7.2.7) is feasible to implement depends on whether the *inverse function of the integral of f(y)* is itself feasible to compute, either analytically or numerically. Sometimes it is, and sometimes it isn't.

Incidentally, (7.2.7) has an immediate geometric interpretation: Since $F(y)$ is the area under the probability curve to the left of y, (7.2.7) is just the prescription: choose a uniform random x, then find the value y that has that fraction x of probability area to its left, and return the value y.

Normal (Gaussian) Deviates

Transformation methods generalize to more than one dimension. If $x_1, x_2, \ldots$ are random deviates with a *joint* probability distribution $p(x_1, x_2, \ldots)$ $dx_1 dx_2 \ldots$, and if $y_1, y_2, \ldots$ are each functions of all the x's (same number of y's as x's), then the joint probability distribution of the y's is

$$p(y_1, y_2, \ldots) dy_1 dy_2 \ldots = p(x_1, x_2, \ldots) \left| \frac{\partial(x_1, x_2, \ldots)}{\partial(y_1, y_2, \ldots)} \right| dy_1 dy_2 \ldots \tag{7.2.8}$$

where $|\partial(\ \ )/\partial(\ \ )|$ is the Jacobian determinant of the x's with respect to the y's (or reciprocal of the Jacobian determinant of the y's with respect to the x's).

Law of large numbers

Coin tossing $S = (\underset{s_1}{\text{head}}, \underset{s_2}{\text{tail}})$

$$P(s_1) = P(s_2) = \frac{1}{2} \quad \text{fair coin} \qquad \mu(s_1) = \mu(s_2) = 1.$$

$$\lambda(s) = +1 \quad \text{if } s = s_1$$
$$= -1 \quad \text{if } s = s_2$$

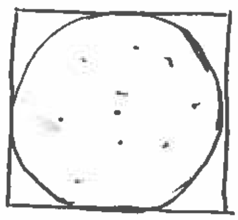
$$E(\lambda) = \frac{1}{2} - \frac{1}{2} = 0$$

Calculate $E_N(\lambda)$ for chosen N.

$$\lim_{N \to \infty} E_N(\lambda) = E(\lambda) \longleftarrow \text{central limit theorem}$$

It's all about how to limit $[E_N(\lambda) - E(\lambda)]^2$

Or, determine the value of π



π = proportion of accepted points

$S = [0,1] \times [0,1] \qquad d\mu = dx\,dy = dS$

$\lambda(s) \sim 1$ within the circle

~ 0 outside the "

$P(s) = 1$ All we'll have to do is to draw from the distribution many times

What happens when we cannot sample from a probability distribution?

We would like to sample from the probability distribution $P(x)$

$$E(\lambda) = \int_S \lambda(s)\, P(s)\, ds = \int_S \lambda(s) \frac{P(s)}{Q(s)} Q(s)\, ds$$

$$= \int_S \underbrace{\frac{\lambda(s) P(s)}{Q(s)}}_{\lambda'(s)} Q(s)\, ds$$

$$= \int_S \lambda'(s)\, Q(s)\, ds$$

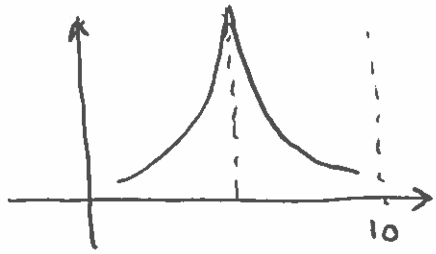
Principle of <u>importance sampling</u>

The trick is to have $\lambda'(s)$ not to get large; means that the regions of s where P is large, we should also choose Q large.

or <u>diminishing return</u>

$$\int_0^{10} \exp(-2|x-5|)\,dx$$

$S = (0, 10)$; $P(s) = 1$ $\lambda(x) = \exp(-2|x-5|)$



Choose $Q(s) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-5)^2}{2}\right)$

<u>stratified sampling</u>

Break S into "strata", i.e. different smaller domains, S_i

evaluate $E(\lambda_i) \longrightarrow E(\lambda)$