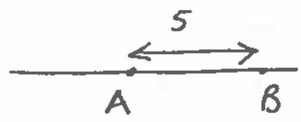
Alice and Bob live in one dimension: a line

How far is Alice from Bob $A, B \in \mathbb{R}^1$

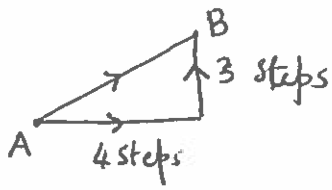


- Vectors
- Co-ordinate Systems (Basis)
- Primitives
- Projections

$\boxed{2D}$

5 is then a scalar,

Alice and Bob live on the space of the blackboard. which is two dimensional



4 steps to the east
then 3 " " " north } to reach Bob from Alice

So you need two numbers:

Also called a tuplet of numbers $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

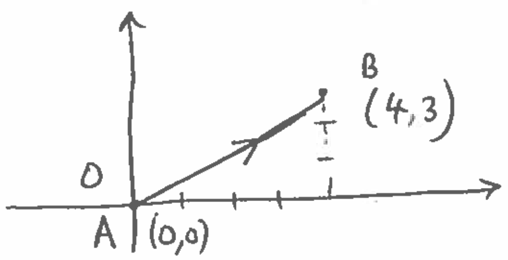
Now, this quantity $\vec{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is a vector.

$\vec{v} \equiv v$: I'm lazy not to draw the arrow.

We are in a computer science class, and not on a field. So we say that 4 units in x-direction and 3 units in y direction. We use Cartesian co-ordinates system.

This Immediately gives us the notion of an origin.

To reach Bob, wrt. Alice, move 4 units in x and 3 units in y



This is now my co-ordinate system.

These points are members of $\mathbb{R}^2$
$A, B \in \mathbb{R}^2$

If I'm in 3d, then a point requires a 3-tuplet or triplet $(3.5, 4\frac{1}{3}, \pi)$ to describe. This a point that is a member of $\mathbb{R}^3$

Similarly $(3, 4, 5, 6) \in \mathbb{R}^4$
⋮
etc.

So vectors and co-ordinate systems are very much intertwined.

$(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ v_n \end{pmatrix}$ is a vector spanning the origin to the location of the point.

$$\left.\begin{array}{l} v_x : \text{ x-component of the vector } \bar{v} \\ v_y : \text{ y-component of the vector } \bar{v} \end{array}\right\} \text{In } \mathbb{R}^2$$

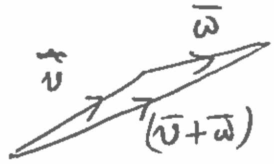
$$\left.\bar{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \xleftarrow{} \text{z-component of vector } \bar{v} \right\} \text{In } \mathbb{R}^3$$

Example 0 on page 7

Similarly

$$\left.\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_k \\ v_d \end{pmatrix} \xleftarrow{} \text{k-th component of vector } \bar{v} \right\} \text{In } \mathbb{R}^d$$

Properties of vectors:



Addition:

$\mathbb{R}^2$: $\begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5+3 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$

$$\bar{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \quad \bar{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix}; \quad \bar{v}+\bar{\omega} = \begin{pmatrix} v_x+\omega_x \\ v_y+\omega_y \end{pmatrix} = \bar{\omega}+\bar{v}$$

Addition is associative:

$$\bar{a} + (\bar{b}+\bar{c}) = (\bar{a}+\bar{b}) + \bar{c}$$

Scalar multiplication:

$$\lambda\bar{v} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_d \end{pmatrix}$$

Null vector or zero vector

all elements or all components are zero: $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Length of a vector:

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}$$ only for Cartesian co-ordinate system

Unit vector $\hat{v} = \frac{\bar{v}}{\|\bar{v}\|}$; $\lambda = \frac{1}{\|\bar{v}\|}$

Length of $\hat{v}$ is unity.

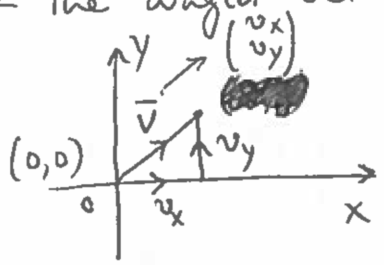
$$\hat{v} = \begin{pmatrix} \frac{v_1}{\|\bar{v}\|} \\ \frac{v_2}{\|\bar{v}\|} \\ \vdots \\ \frac{v_d}{\|\bar{v}\|} \end{pmatrix}$$

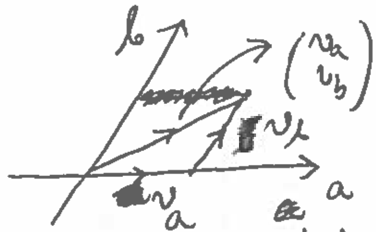
$\hat{x}$: unit vector in x-direction
$\hat{y}$: " " " y-direction
and so on...

Basis (pl. Bases)

Choice of reference directions.

Cartesian co-ordinate system: the bases are $(\hat{x}, \hat{y}, \hat{z} \ldots)$ orthogonal basis and orthonormal basis
The angles between any two basis ~~directions~~ vectors is $90°$.





Two vectors that are parallel/antiparallel are called linearly dependent vectors.

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} ; \quad \lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_d \end{pmatrix}$$

If two vectors v_1 & v_2 can be related as $v_2 = \lambda v_1$ are linearly dependent.

In d-dimension any set of d linearly independent vectors can be used to form the basis vectors.

You can reach any point in d-dimensions using these basis vectors. We'll see applications of non-Cartesian basis vectors later.

Multiplication with a vector

Scalar product or dot product

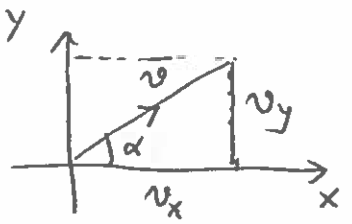
$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}, \quad \bar{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_d \end{pmatrix}$$

$$\bar{v} \cdot \bar{\omega} = \bar{\omega} \cdot \bar{v} = \underbrace{v_1 \omega_1 + v_2 \omega_2 + \cdots + v_d \omega_d}_{\text{Scalar quantity}}$$

- $\bar{v} \cdot \bar{\omega} = 0 \Rightarrow \bar{v}$ and $\bar{\omega}$ are orthogonal to each other $\bar{v} \perp \bar{\omega}$
- $\hat{v} \cdot \hat{\omega} = 0 \Rightarrow \hat{v}$ and $\hat{\omega}$ are orthonormal to each other.

→ $\bar{v} \cdot \bar{\omega}$ means projection of the vector $\bar{v}$ on to vector $\bar{\omega}$.

2D: $\bar{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ ← x-component, ← y-component



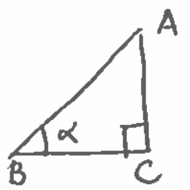
$$v_x = \bar{v} \cdot \hat{x} = \|\bar{v}\| \cos\alpha$$
$$v_y = \bar{v} \cdot \hat{y} = \|\bar{v}\| \sin\alpha$$

E.g. $v_x = \bar{v} \cdot \hat{x} = 0$ means $\cos\alpha = 0$ $\alpha = 90^\circ$

→ $\bar{v} \cdot \bar{\omega} = v\, \omega \cos\alpha$: α is the angle between $\bar{v}$ and $\bar{\omega}$.

$$\Rightarrow \cos\alpha = \frac{\bar{v} \cdot \bar{\omega}}{v\, \omega} = \frac{\bar{v} \cdot \bar{\omega}}{\|\bar{v}\| \, \|\bar{\omega}\|}$$

Some elementary trigonometry



$$\frac{AC}{AB} = \sin\alpha$$
$$\frac{BC}{AB} = \cos\alpha$$
$$\frac{AC}{BC} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

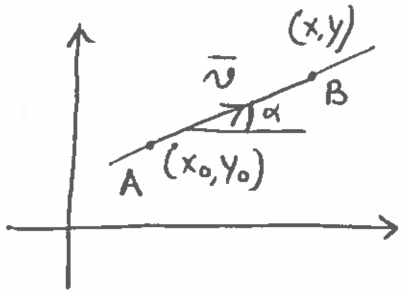
Pythagoras: $AC^2 + BC^2 = AB^2$

$\sin^2\alpha + \cos^2\alpha = 1$

α	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
0	0	1	0
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

Primitives and projections in 2D:

Primitives are basically objects/shapes/forms.
Such as line, circle, point etc.



Lines: back to Alice and Bob story.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\hat{v}$$

$\boxed{\text{Example 1: page 7}}$

<u>parametric form</u>

$\hat{v}$ is the unit vector $\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \end{pmatrix}$.

<u>Note</u>: that in the parametric form you do not have to use unit vector $\hat{v}$. You can also use $\bar{v}$ itself. However, if you use the unit vector $\hat{v}$, then $|t|$ really is the length of the line connecting (x_0, y_0) and (x, y).

$$x = x_0 + t v_x \qquad t = \frac{x - x_0}{v_x} = \frac{y - y_0}{v_y}$$
$$y = y_0 + t v_y \; ;$$

$$y - y_0 = \frac{v_y}{v_x}(x - x_0)$$

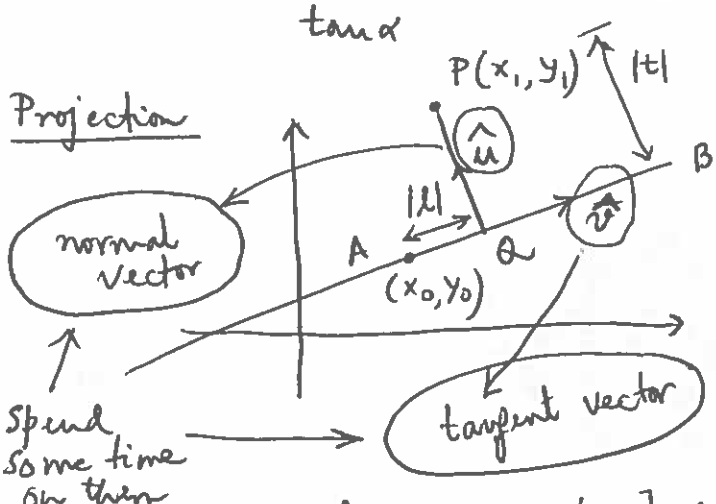
$$y = \underbrace{\frac{v_y}{v_x}}_{m} x + \underbrace{y_0 - \frac{v_y}{v_x} x_0}_{c}$$

$$= \underset{\tan\alpha}{m} x + c \quad : \quad \text{slope-intercept form}$$

<u>Implicit form</u>

$$Ax + By + C = 0$$
$$y = \underbrace{-\frac{A}{B}}_{m} x \underbrace{- \frac{C}{B}}_{=c}$$

<u>Projection</u>



$$\hat{u} \cdot \hat{v} = 0 \qquad \hat{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad v_x^2 + v_y^2 = 1$$

$$\hat{u} = \begin{pmatrix} v_y \\ -v_x \end{pmatrix}$$

$\boxed{\text{: Example 2 page 7}}$

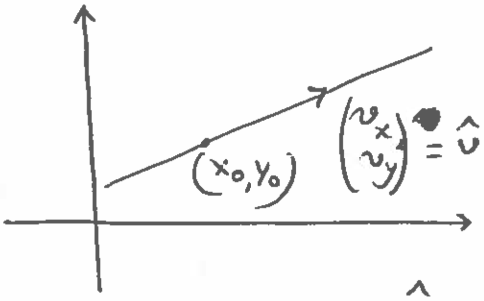
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + l \begin{pmatrix} v_x \\ v_y \end{pmatrix} \quad : AB$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} v_y \\ -v_x \end{pmatrix} \quad : PQ.$$

$$x_0 + l v_x = x_1 + t v_y \,]\times v_y$$
$$y_0 + l v_y = y_1 - t v_x \,]\times v_x$$

$$x_0 v_y - y_0 v_x = x_1 v_y - y_1 v_x + t$$

$$t = (x_0 - x_1) v_y - (y_0 - y_1) v_x$$



$$\hat{u} = \begin{pmatrix} -v_y \\ v_x \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \ell \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\hat{u} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = 0$$

$$\cancel{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \ell \begin{pmatrix} v_x \\ v_y \end{pmatrix}}$$

$$\cancel{y = y_0 + \ell v_y}$$

$$\cancel{x = x_0 + \ell v_x}$$

At Q: $y_2 = m x_2 + c$

At P: $y_1 - (m x_1 + c)$

$$Z = y_1 - y_2 - (m x_1 + c) + (m x_2 + c)$$

$$Z = y_1 - y_2 - m(x_1 - x_2) \,; \quad m = \frac{v_y}{v_x}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + t \begin{pmatrix} v_y \\ -v_x \end{pmatrix}$$

$$x_1 = x_2 + t v_y$$

$$y_1 = y_2 - t v_x$$

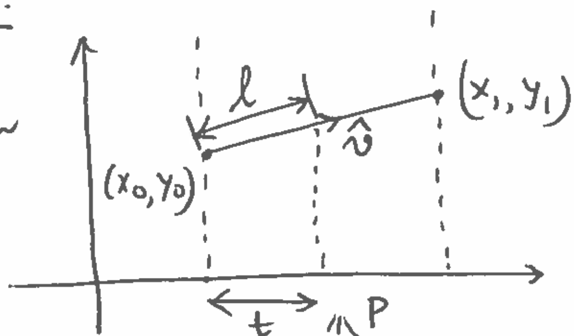
$$Z = y_1 - y_2 - \frac{v_y}{v_x}(x_1 - x_2)$$

$$= -t v_x - v_y t v_y / v_x$$

$$= -\frac{t}{v_x}\left(v_x^2 + v_y^2\right).$$

<u>Projections ...</u>

Parallel projection

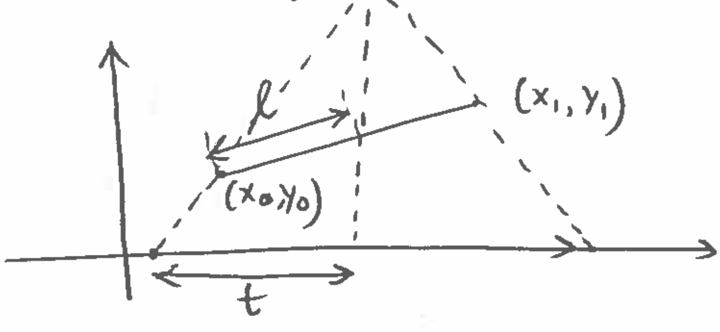


$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \hat{v} \ell$$

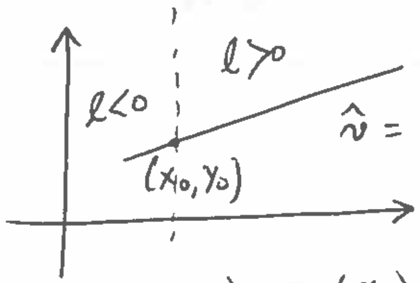
$$x = x_0 + v_x \ell = x_0 + t$$

$$t = v_x \ell$$

Perspective Projection



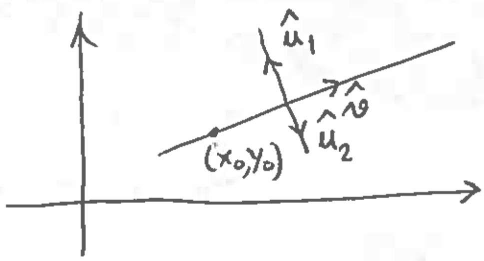
Example 1:



$$\hat{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \begin{matrix} = \frac{3}{5} \\ = \frac{4}{5} \end{matrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \ell \begin{pmatrix} v_x \\ v_y \end{pmatrix} \qquad \ell \in (-\infty, \infty)$$

Example 2:



$$\hat{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \begin{matrix} \leftarrow \frac{3}{5} \\ \leftarrow \frac{4}{5} \end{matrix}$$

$$\hat{u}_2 = \begin{pmatrix} v_y \\ -v_x \end{pmatrix} \begin{matrix} \leftarrow \frac{4}{5} \\ \leftarrow -\frac{3}{5} \end{matrix}$$

or $$\hat{u}_1 = \begin{pmatrix} -v_y \\ v_x \end{pmatrix} = \begin{pmatrix} -4/5 \\ \frac{3}{5} \end{pmatrix}$$

Example 0:

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \quad : \quad \text{in 2D} \quad \bar{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} \begin{matrix} \leftarrow \text{x-component} \\ \end{matrix}$$

$v_x > 0$	points	to	$+x$-axis
< 0	"	"	$-x$-axis
$v_y > 0$	"	"	$+y$-axis
< 0	"	"	$-y$-axis