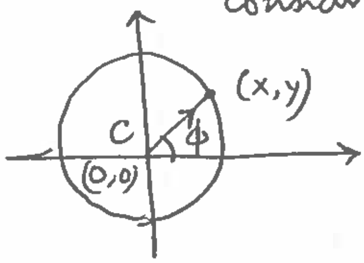
Further primitives in 2D, and co-ordinate systems.

Consider a circle



$$\bar{V} = \begin{pmatrix} x \\ y \end{pmatrix}$$

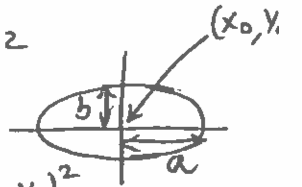
$$x^2 + y^2 = r^2$$

with centre ~~at~~ @ (x_0, y_0) : $(x-x_0)^2 + (y-y_0)^2 = r^2$

or in the form $f(x,y) = 0$

$$(x-x_0)^2 + (y-y_0)^2 - r^2 = 0$$

Ellipse



$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

$x = x_0 + r\cos\phi$

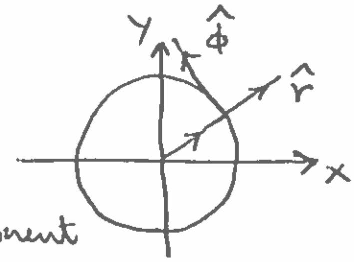
$y = y_0 + r\sin\phi$;

$0 = 0° \leqslant \phi < 360° = 2\pi$

circular co-ordinate system
we need (x_0, y_0) and $\underline{\phi}$

For the co-ordinate system there are two special unit vectors

Cartesian	circular
$\hat{x}, \hat{y}$	$\hat{r}, \hat{\phi}$



$\hat{r}\cdot\hat{x} = \cos\phi$

$\hat{r}\cdot\hat{y} = \sin\phi$

x-component

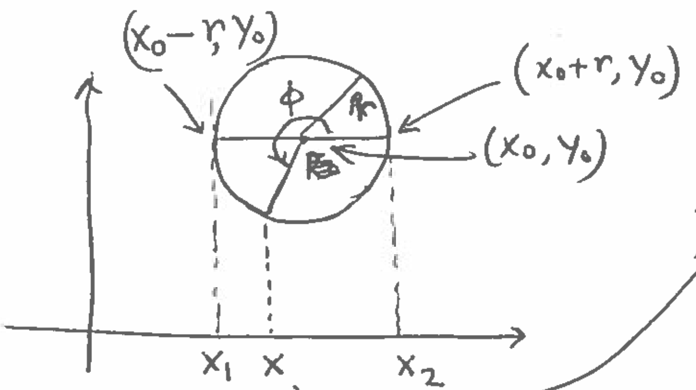
$$\hat{r} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

y-component

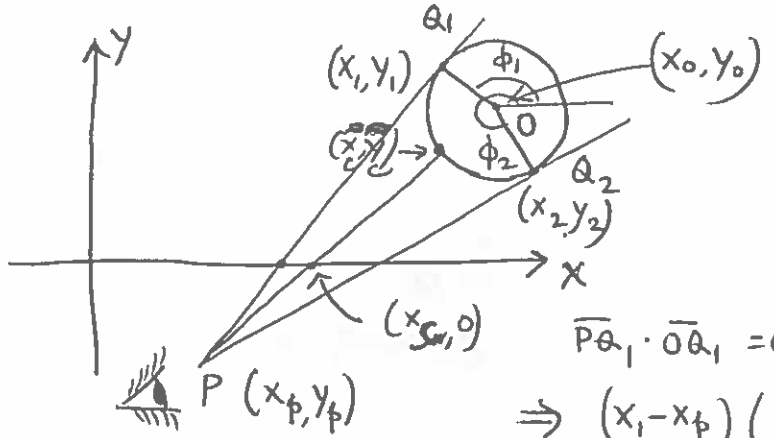
$$\hat{r} = (\hat{r}\cdot\hat{x})\,\hat{x} + (\hat{r}\cdot\hat{y})\,\hat{y}$$

$$\hat{\phi} = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \qquad \hat{\phi} = (\hat{\phi}\cdot\hat{x})\,\hat{x} + (\hat{\phi}\cdot\hat{y})\,\hat{y}$$

projection



$x = x_0 + r \cos\phi$



Determine (x_1, y_1) & (x_2, y_2)

$$\overline{PQ_1} = \begin{pmatrix} x_1 - x_p \\ y_1 - y_p \end{pmatrix} \quad \overline{OQ_1} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}$$

$$\overline{PQ_1} \cdot \overline{OQ_1} = 0$$

$$\Rightarrow (x_1 - x_p)(x_1 - x_0) + (y_1 - y_p)(y_1 - y_0) = 0$$

$$\Rightarrow (x_1 - x_0 + x_0 - x_p)(x_1 - x_0) + (y_1 - y_0 + y_0 - y_p)(y_1 - y_0) = 0$$

$$\Rightarrow \underbrace{(x_1 - x_0)^2 + (y_1 - y_0)^2}_{r^2} + \underbrace{(x_1 - x_0)}_{r\cos\phi_1}(x_0 - x_p) + \underbrace{(y_1 - y_0)}_{r\sin\phi_1}(y_0 - y_p) = 0$$

$$r\cos\phi_1 (x_0 - x_p) + r\sin\phi_1 (y_0 - y_p) + r^2 = 0$$

$$\Rightarrow \left.\begin{array}{l} \cos\phi_1 (x_0 - x_p) + \sin\phi_1 (y_0 - y_p) + r = 0 \\ \sin^2\phi_1 + \cos^2\phi_1 = 1 \end{array}\right\} \text{solve } \phi_1 \downarrow (x_1, y_1)$$

Similarly, solve $\phi_2 \rightarrow (x_2, y_2)$

<u>You'll be able to see all $\phi_1 \leq \phi \leq \phi_2$</u>

$$\frac{y - y_c}{x - x_c} = \frac{y - y_p}{x - x_p}$$

Screen: $y = 0 \Rightarrow (x - x_p) y_c = y_p (x - x_c)$

$$x = \frac{x_p y_c - x_c y_p}{-y_p + y_c}$$

$$\boxed{x = \frac{x_p (y_0 + r\sin\phi) - (x_0 + r\cos\phi) y_p}{y_0 + r\sin\phi - y_p}}$$

$$\phi_1 \leq \phi \leq \phi_2$$

We move on to 3 dimensions:

$\hat{x}$, $\hat{y}$ and $\hat{z}$

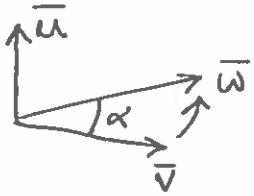
$$\bar{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \qquad \bar{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$\bar{v}$, $\bar{w}$: cross-product

$$\bar{u} = \bar{v} \times \bar{w} = \begin{pmatrix} v_y w_z - v_z w_y \\ v_z w_x - v_x w_z \\ v_x w_y - v_y w_x \end{pmatrix}$$

$\bar{v} \cdot \bar{u} = \bar{u} \cdot \bar{w} = 0$ so, $\bar{u} \perp \bar{v}$, $\bar{u} \perp \bar{w}$

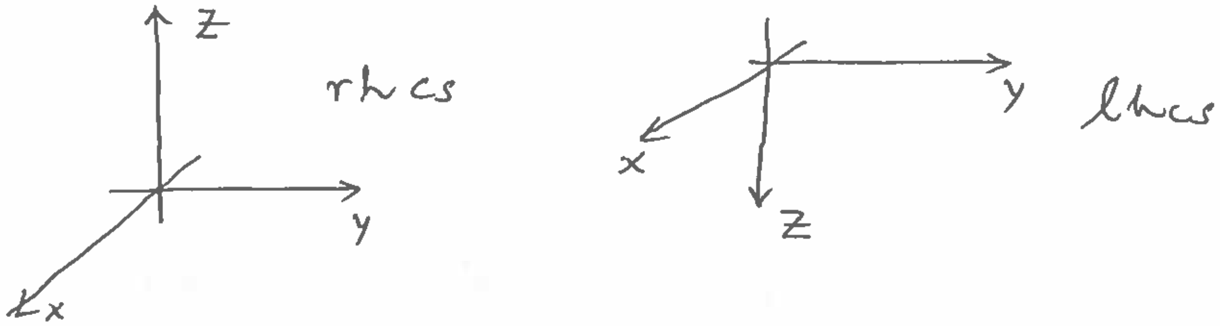
$\bar{u}$ is orthogonal to both $\bar{v}$ and $\bar{w}$.



$$u = \|\bar{u}\| = \|\bar{v}\| \, \|\bar{w}\| \sin\alpha$$
$$= v w \sin\alpha$$

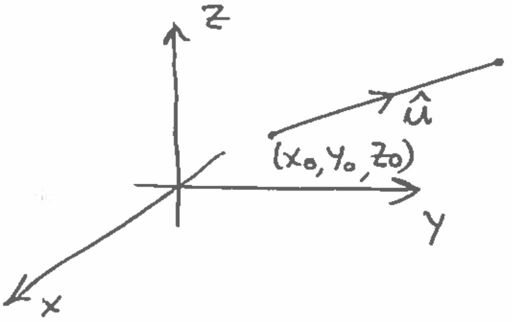
Right-handed co-ordinate system: $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$

Left-handed ,, ,, : $\hat{x} \times \hat{y} = -\hat{z}$ $\hat{y} \times \hat{z} = -\hat{x}$, $\hat{z} \times \hat{x} = -\hat{y}$



choose rhcs to work with.

Eqn. of a line in 3d

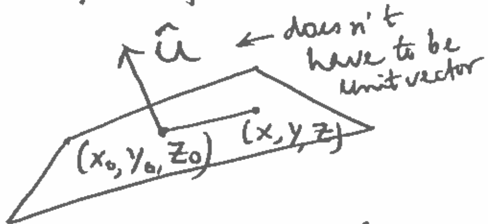


$$\hat{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

$$x - x_0 = l u_x$$
$$y - y_0 = l u_y$$
$$z - z_0 = l u_z$$

$$\boxed{\frac{x - x_0}{u_x} = \frac{y - y_0}{u_y} = \frac{z - z_0}{u_z}}$$

Eqn. of a plane in 3d.



$$(x - x_0) u_x + (y - y_0) u_y + (z - z_0) u_z = 0$$

$$u_x x + u_y y + u_z z = u_x x_0 + u_y y_0 + u_z z_0$$

$$Ax + By + Cz + D = 0$$

Recall in 2D: $Ax + By + C = 0$

Similarly in 4D: $Ax + By + Cz + Dw + E = 0$

$\vdots$

and so on

$$u_x = \frac{A}{\sqrt{A^2+B^2+C^2}}, \quad u_y = \frac{B}{\sqrt{A^2+B^2+C^2}}, \quad u_z = \frac{C}{\sqrt{A^2+B^2+C^2}}$$

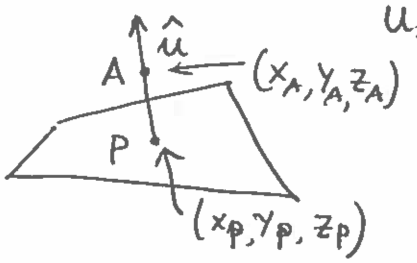
Also $\longrightarrow$ $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} + l \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Two unit basis vectors

Projections on a plane:

1. A point

Plane eqn. $Ax + By + Cz + D = 0$

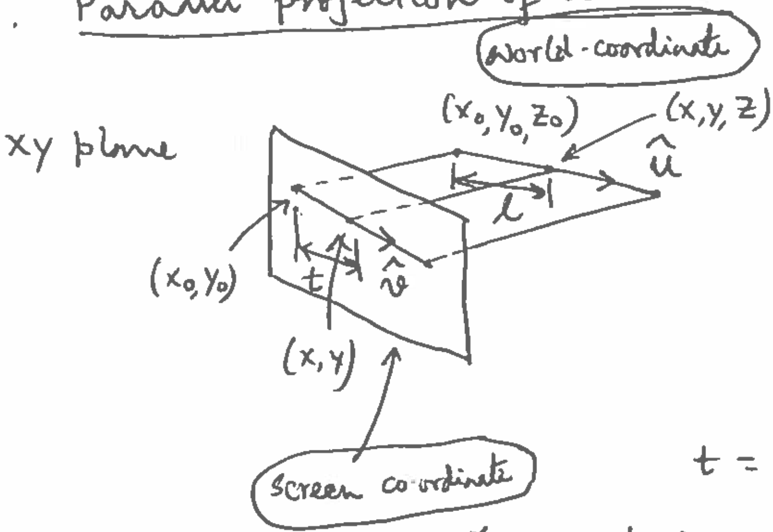


u_x, u_y, u_z are obtained from A, B, C

$$\left.\begin{aligned} x_A &= x_p + t u_x \\ y_A &= y_p + t u_y \\ z_A &= z_p + t u_z \\ A x_p + B y_p &+ C z_p + D = 0 \end{aligned}\right\}$$

Four equations for four unknowns x_p, y_p, z_p, t can be solved.

2. Parallel projection of a line



$$x = x_0 + l u_x$$
$$y = y_0 + l u_y$$
$$z = z_0 + l u_z$$

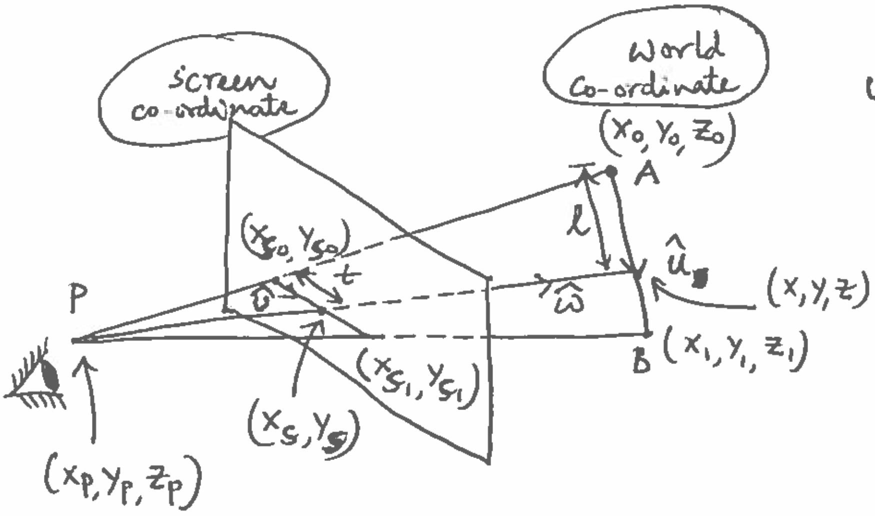
$$t^2 = (x - x_0)^2 + (y - y_0)^2$$
$$= l^2 (u_x^2 + u_y^2)$$

$$t = l \sqrt{u_x^2 + u_y^2}$$

$$x - x_0 = t v_x, \quad y - y_0 = t v_y; \quad v_x^2 + v_y^2 = 1$$

$$v_x = \frac{u_x}{\sqrt{u_x^2 + u_y^2}}, \quad v_y = \frac{u_y}{\sqrt{u_x^2 + u_y^2}}$$

3. Perspective projection of a line



$$u_x = \frac{x_1 - x_0}{\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2}}$$

$$\vdots$$

$$\left.\begin{aligned} x - x_0 &= l u_x \\ y - y_0 &= l u_y \\ z - z_0 &= l u_z \end{aligned}\right\} \quad ①$$

$$\left.\begin{aligned} x_s - x_{s0} &= t v_x \\ y_s - y_{s0} &= t v_y \\ z_s &= 0 \end{aligned}\right\} \quad ② \qquad \left.\begin{aligned} x - x_p &= r \omega_x \\ y - y_p &= r \omega_y \\ z - z_p &= r \omega_z \end{aligned}\right\} \quad ③$$

① into ③

$$\left.\begin{aligned} x_0 + l u_x - x_p &= r \omega_x \\ y_0 + l u_y - y_p &= r \omega_y \\ z_0 + l u_z - z_p &= r \omega_z \end{aligned}\right\} \quad ⑤ \qquad \left.\begin{aligned} x_s - x_p &= q \omega_x \\ y_s - y_p &= q \omega_y \\ \underbrace{z_s}_{=0} - z_p &= q \omega_z \end{aligned}\right\} \quad ④$$

~~② into ④~~ ④ + ⑤

$$\Rightarrow \frac{x_s - x_p}{x_0 + l u_x - x_p} = \frac{y_s - y_p}{y_0 + l u_y - y_p} = -\frac{z_p}{z_0 + l u_z - z_p} = c_l$$

$$\boxed{\begin{aligned} x_s &= x_p + c_l (x_0 + l u_x - x_p) \\ y_s &= y_p + c_l (y_0 + l u_y - y_p) \end{aligned}}$$

$$c_0 = -\frac{z_p}{z_0 - z_p}$$

$$\boxed{\begin{aligned} x_{s0} &= x_p + c_0 (x_0 - x_p) \\ y_{s0} &= y_p + c_0 (y_0 - y_p) \end{aligned}}$$

$$X_s - X_{s_0} = C_\ell \left(x_0 + \ell u_x - x_p\right) - c_0 \left(x_0 - x_p\right)$$

$$= \frac{z_p (x_0 - x_p)}{z_0 - z_p} - \frac{z_p (x_0 + \ell u_x - x_p)}{z_0 + \ell u_z - z_p}$$

$$= \frac{\cancel{z_p (x_0 - x_p)(z_0 - z_p)} + \ell u_z z_p (x_0 - x_p) - \cancel{z_p (x_0 - x_p)(z_0 - z_p)} - \ell u_x z_p (z_0 - z_p)}{(z_0 - z_p)(z_0 + \ell u_z - z_p)}$$

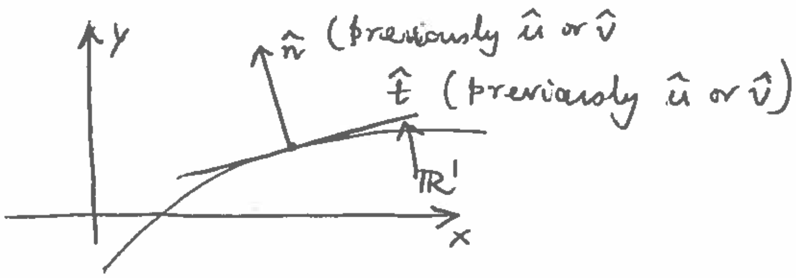
$$\Rightarrow \boxed{\begin{aligned} X_s - X_{s_0} &= \frac{\ell z_p \left[u_z (x_0 - x_p) - u_x (z_0 - z_p)\right]}{(z_0 - z_p)(z_0 + \ell u_z - z_p)} \\ Y_s - Y_{s_0} &= \frac{\ell z_p \left[u_z (y_0 - y_p) - u_y (z_0 - z_p)\right]}{(z_0 - z_p)(z_0 + \ell u_z - z_p)} \end{aligned}}$$

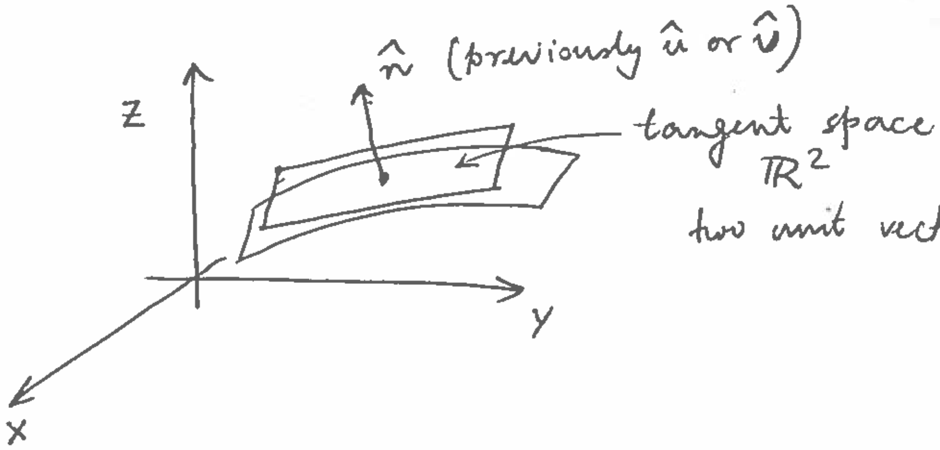
Case: If the line is parallel to the plane, then $u_z = 0$

$$\frac{x_s - x_p}{x - x_p} = \frac{y_s - y_p}{y - y_p} = \frac{\cancel{z_s} - z_p}{z - z_p}$$

$$\boxed{\begin{aligned} X_s &= \frac{(x - x_p)(-z_p)}{z - z_p} + x_p = \frac{z x_p - x z_p}{z - z_p} \\ Y_s &= \frac{z y_p - y z_p}{z - z_p} \end{aligned}}$$

Tangent space and normal space





two unit vectors $\hat{t}$ & $\hat{b}$.

bitangent

Projecting a plane to the screen

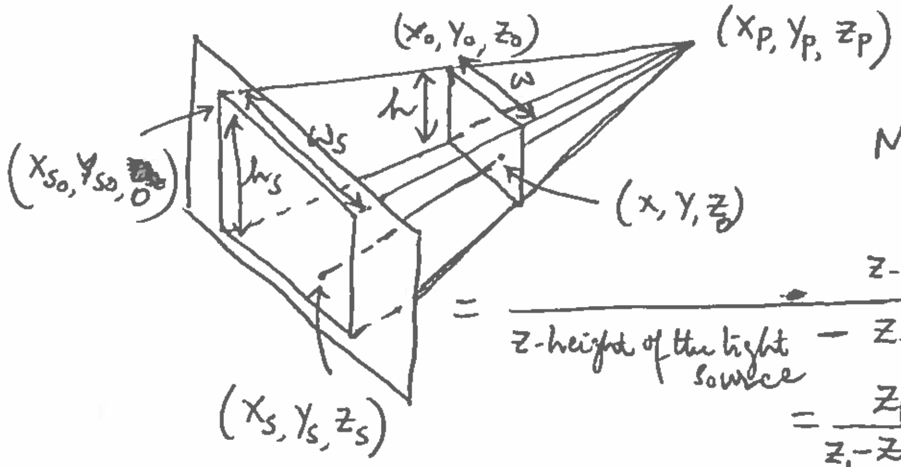


Image gets magnified

Magnification M

$$= \frac{\text{z-height of the light source}}{\text{z-height of the light source} - \text{z-height of the plane}}$$

$$= \frac{z_p}{z_p - z_0} = \frac{h_s}{h} = \frac{w_s}{w}$$

So, if you know (x_{s0}, y_{s0}), w_s and h_s then you know everything about this projection, i.e. where does a point (x, y) in the world co-ordinate projects to the screen coordinates (x_s, y_s).

In particular, $(y_{s0} - y_p) = M\,(y_0 - y_p)$

Similarly $(x_{s0} - x_p) = M\,(x_0 - x_p)$