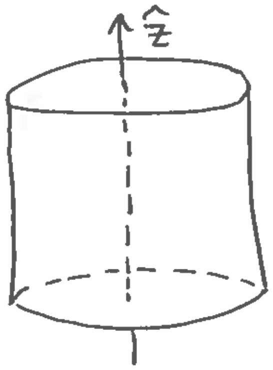
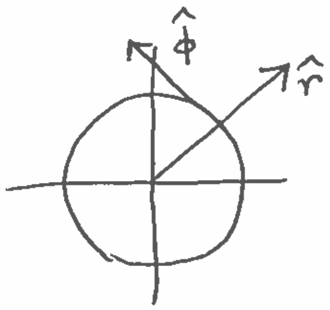
Other co-ordinate systems in 3d

Cylindrical co-ordinate system:



$$\left.\begin{aligned} & r,\ \phi,\ z \\ & x = r\cos\phi \\ & y = r\sin\phi \end{aligned}\right\}$$ extension of circular co-ordinate system



surface of a cylinder needs two co-ordinates: ϕ and z

Spherical co-ordinate system:

(r, θ, ϕ)

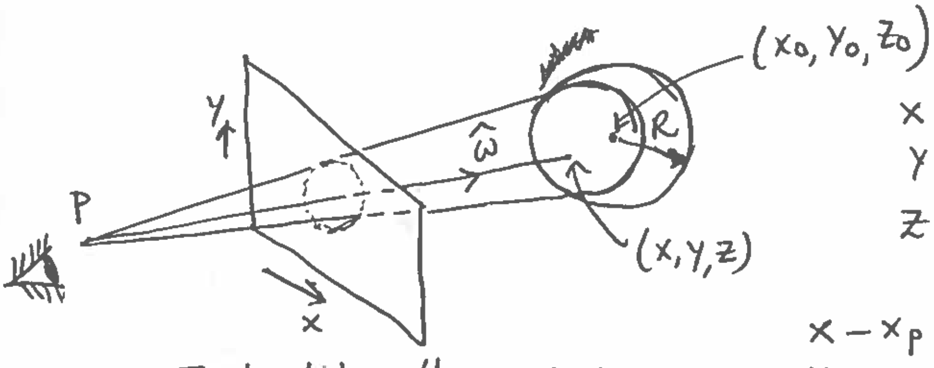
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

Need picture

Show $\hat{r}, \hat{\theta}, \hat{\phi}$ unit vectors!

surface of a sphere also needs two co-ordinates: θ and ϕ

Projection of a sphere on to a screen



$$x = x_0 + R\sin\theta\cos\phi$$
$$y = y_0 + R\sin\theta\sin\phi$$
$$z = z_0 + R\cos\theta$$

$$\left.\begin{aligned} x - x_p &= r\,\omega_x \\ y - y_p &= r\,\omega_y \\ z - z_p &= r\,\omega_z \end{aligned}\right\} \qquad \left.\begin{aligned} x_s - x_p &= q\,\omega_x \\ y_s - y_p &= q\,\omega_y \\ -z_p &= q\,\omega_z \end{aligned}\right\}$$

Just like the moon, except that we are ~~very~~ 220 moon radius away.

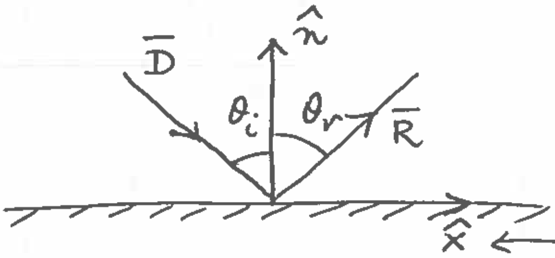
$$\frac{x_s - x_p}{x_0 + R\sin\theta\cos\phi - x_p} = \frac{y_s - y_p}{y_0 + R\sin\theta\sin\phi - y_p} = \frac{-z_p}{z_0 + R\cos\theta - z_p} = c_\theta$$

$$x_s = x_p + c_\theta\left(x_0 + R\sin\theta\cos\phi - x_p\right)$$

$$y_s = y_p + c_\theta\left(y_0 + R\sin\theta\sin\phi - y_p\right)$$

Properties of light : no werkcollege exercises, but . . .

Reflection



$\theta_i = \theta_r$

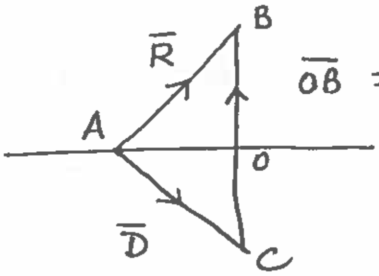
$\hat{t}$ (tangent space to the surface)

$$\bar{D} = (\bar{D}\cdot\hat{x})\,\hat{x} + (\bar{D}\cdot\hat{n})\,\hat{n}$$

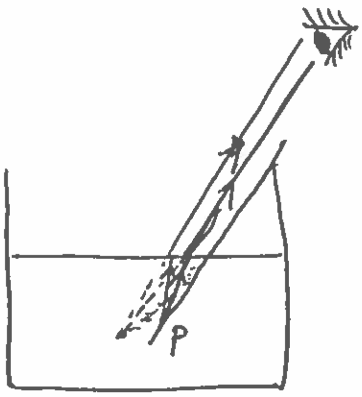
$$\underbrace{(\bar{D}\cdot\hat{n})\,\hat{n}}_{\bar{D}_\perp} = \bar{D} - (\bar{D}\cdot\hat{x})\,\hat{x} \;;\quad \underbrace{(\bar{D}\cdot\hat{x})\,\hat{x}}_{\bar{D}_\parallel} = \bar{D} - (\bar{D}\cdot\hat{n})\,\hat{n}$$

$$\bar{R} = (\bar{D}\cdot\hat{x})\,\hat{x} - (\bar{D}\cdot\hat{n})\,\hat{n}$$

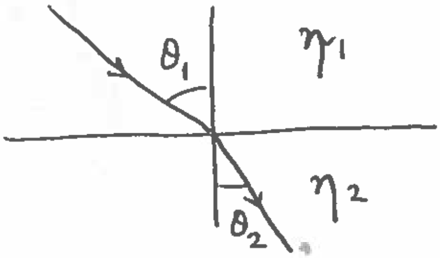
$$= \bar{D} - 2\,(\bar{D}\cdot\hat{n})\,\hat{n}$$



$\overline{OB} = -(\bar{D}\cdot\hat{n})\,\hat{n}$; $\overline{OC} = (\bar{D}\cdot\hat{n})\,\hat{n}$



Refraction



$\eta_2 > \eta_1$

η = refractive index

$\eta_{air} = \eta_{vacuum} = 1.0$

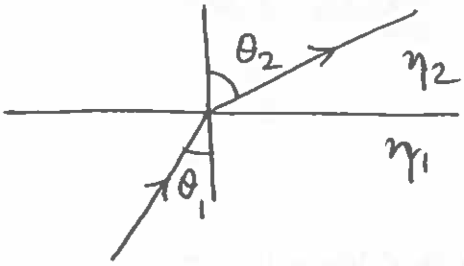
$\eta_{water} \simeq \frac{4}{3} \simeq 1.33$

$\eta_{glass} \simeq 1.5 \simeq \frac{3}{2}$

$$\eta_1 \sin\theta_1 = \eta_2 \sin\theta_2$$

$$\theta_2 = \sin^{-1}\left(\frac{\eta_1}{\eta_2}\sin\theta_1\right)$$

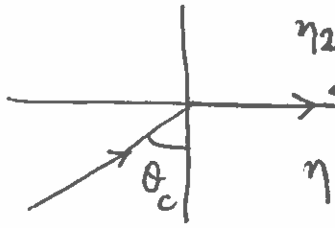
$$\eta_1 < \eta_2 \Rightarrow \theta_2 < \theta_1$$



$$\eta_1 > \eta_2 \Rightarrow \theta_2 > \theta_1$$

But $\theta_{2,max} = 90°$

$$\sin\theta_{2\,max} = 1.0$$

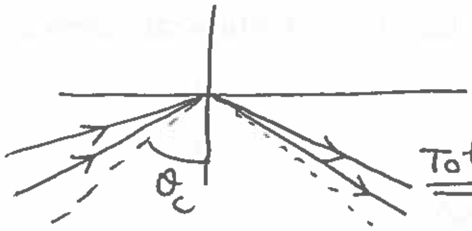


grazing refraction

$$\eta_1 \sin\theta_c = \eta_2$$

$$\theta_c = \sin^{-1}\left(\frac{\eta_2}{\eta_1}\right)$$

what happens if $\theta_i > \theta_c$?

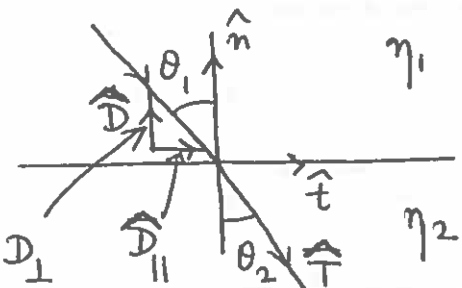


<u>Total</u> internal reflection

$$0 \leq \sin^2\theta_2 = \frac{\eta_1^2}{\eta_2^2}\sin^2\theta_1 \leq 1$$

$$k = 1 - \frac{\eta_1^2}{\eta_2^2}\sin^2\theta_1 = 1 - \frac{\eta_1^2}{\eta_2^2}\left(1 - \cos^2\theta_1\right) \geq 0$$

$k < 0 \Rightarrow$ Total internal reflection.



$$\hat{D}_{\parallel} = (\hat{D}\cdot\hat{t})\,\hat{t} = \sin\theta_1\,\hat{t}$$

$$\hat{D}_{\perp} = \hat{D} - 2(\hat{D}\cdot\hat{t})\,\hat{t} = -\cos\theta_1\,\hat{n}$$

$$\hat{T}_{\perp} = (\hat{T}\cdot\hat{n})\,\hat{n} = -\cos\theta_2\,\hat{n}$$

$$\hat{T}_{\parallel} = \sin\theta_2\,\hat{t}$$

$$\hat{T} = \sin\theta_2\,\hat{t} - \cos\theta_2\,\hat{n}$$

$$= \frac{\eta_1}{\eta_2}\sin\theta_1\,\hat{t} - \sqrt{1 - \frac{\eta_1^2}{\eta_2^2}\sin^2\theta_1}\;\hat{n}$$

$$= \frac{\eta_1}{\eta_2}(\hat{D}\cdot\hat{t})\,\hat{t} - \sqrt{1 - \frac{\eta_1^2}{\eta_2^2}(\hat{D}\cdot\hat{t})^2}\;\hat{n}$$

$$= \frac{\eta_1}{\eta_2}\left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right) - \sqrt{1 - \frac{\eta_1^2}{\eta_2^2}\left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right)^2}\ \hat{n}$$

$$\left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right)^2 = \left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right)\left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right)$$

$$= 1 - 2(\hat{D}\cdot\hat{n})^2 + (\hat{D}\cdot\hat{n})^2$$

$$= 1 - (\hat{D}\cdot\hat{n})^2$$

$$= \frac{\eta_1}{\eta_2}\left(\hat{D} - (\hat{D}\cdot\hat{n})\hat{n}\right) - \sqrt{\left(1 - \frac{\eta_1^2}{\eta_2^2}\right) + \frac{\eta_1^2}{\eta_2^2}(\hat{D}\cdot\hat{n})^2}\ \hat{n}$$

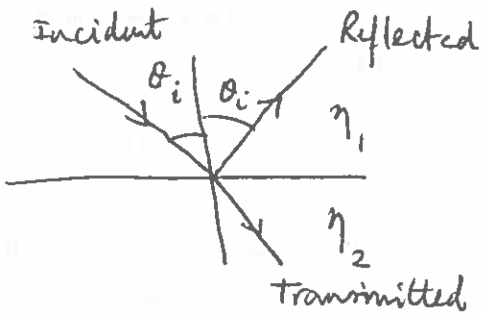
$$= \frac{\eta_1}{\eta_2}\hat{D} + \frac{\eta_1}{\eta_2}\cos\theta_1\,\hat{n} - \sqrt{\left(1 - \frac{\eta_1^2}{\eta_2^2}\right) + \frac{\eta_1^2}{\eta_2^2}\left(1 - \cos^2\theta_1\right)}\ \hat{n}$$

$$\boxed{\hat{T} = \frac{\eta_1}{\eta_2}\hat{D} + \left(\frac{\eta_1}{\eta_2}\cos\theta_1 - \sqrt{k}\right)\hat{n}}$$

Fresnel equations:

Reflectance of unpolarized light:

$$R_r = \underbrace{\left(\frac{\eta_1\cos\theta_i - \eta_2\sqrt{1 - \left(\frac{\eta_1}{\eta_2}\sin\theta_i\right)^2}}{\eta_1\cos\theta_i + \eta_2\sqrt{1 - \left(\frac{\eta_1}{\eta_2}\sin\theta_i\right)^2}}\right)^2}_{\text{Reflectivity}}$$



Schlick's approx. $R_r = R_0 + (1 - R_0)(1 - \cos\theta_i)^5$; $R_0 = \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2$

Transmittance $R_t = 1 - R_r$