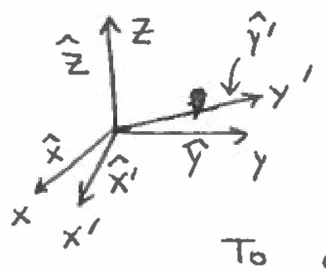
Back to bases



$$\bar{a} = \lambda_1 \hat{x} + \lambda_2 \hat{y} + \lambda_3 \hat{z}$$
$$= \lambda_1' \hat{x}' + \lambda_2' \hat{y}' + \lambda_3 \hat{z}$$

How are λ_1' and λ_2' related to λ_1 and λ_2?

To determine things like this we need to invest in matrices

A vector in d dimensions is a d-tuple of numbers or scalar variables.

$$\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ in } 3d$$

An $m \times n$ matrix is an array of mn scalar values sorted in m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad m = n = 2$$

(matrix dimension is $m \times n$)

a_{ij} are the matrix elements or matrix coefficients

i is the row number, j is the column number

$m = n$ is called a square matrix

Addition of matrices

$$A = \{a_{ij}\} \quad B = \{b_{ij}\} \qquad A + B = \{a_{ij} + b_{ij}\}$$

You can only add matrices of the same dimensions.

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 2-1 & 3+2 \\ 5+2 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 7 & 2 \end{pmatrix}$$

Multiplication by a scalar

like vectors.

$$A = \{a_{ij}\} \qquad B = \lambda A = \{b_{ij}\} = \{\lambda a_{ij}\}$$
$$\Rightarrow b_{ij} = \lambda a_{ij}$$

$$2 \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 10 \\ 8 & 6 \end{pmatrix}$$

Multiplication by matrices

A is a matrix of dimension $m \times n$
B is a matrix of dimension $n \times k$

$C = AB$ is a matrix of dimension $m \times k$

$$c_{ij} = \sum_{j=1}^{n} a_{ij} \, b_{jk}$$

$$\begin{array}{cc} & \xrightarrow{\text{3 columns}} \\ C = \downarrow \text{2 rows} & \underbrace{\begin{pmatrix} 2 & 6 & 1 \\ 5 & 2 & 4 \end{pmatrix}}_{2\times 3} \end{array} \underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}_{3\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$c_{11} = \sum_{j=1}^{3} a_{1j} \, b_{j1} = 2\times 1 + 6\times 2 + 1\times 3 = 17$$

$$c_{12} = \sum_{j=1}^{3} a_{1j} \, b_{j2} = 2\times 4 + 6\times 5 + 1\times 6 = 44$$

$$c_{21} = \sum_{j=1}^{3} a_{2j} \, b_{j1} = 5\times 1 + 2\times 2 + 4\times 3 = 21$$

$$c_{22} = \sum_{j=1}^{3} a_{2j} \, b_{j2} = 5\times 4 + 2\times 5 + 4\times 6 = 54$$

Special matrices

Diagonal matrix

$$A = \begin{pmatrix} 1.5 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$$

Identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Zero matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \emptyset$$

$$\begin{pmatrix} \boxed{2 \quad 6 \quad 1} \\ 5 \quad 2 \quad 4 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}} & \begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \end{pmatrix}$$

dot product between a row and a column!

distributive: $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

Associative: $(AB)C = A(BC)$

$AB \neq BA$!!

Multiplication by special matrices

$\emptyset A = A\emptyset = \emptyset$

$AI = IA = A$

Some further operations/properties/definitions for matrices

Transpose of a matrix A is A^T

row elements

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

Column element

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \end{pmatrix}$$

square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

Determinant of a matrix

$A = -5$ 1×1 matrix

Only defined for square matrices

$\det A = |A| = 5$

notation: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

or, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

How do we calculate it?

define co-factors: Cofactor of an element a_{ij} is ~~the~~ obtained by

1) determinant of the $(n-1)\times(n-1)$ matrix when i & j row + column of the matrix are excluded
2) multiplied by $(-1)^{i+j}$

Example $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

cof. $a_{11} = a_{22}$
" $a_{12} = -a_{21}$
" $a_{21} = -a_{12}$
" $a_{22} = a_{11}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \text{Cof. } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\text{"} \quad a_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Laplace's expansion for determinant

Take any row:

~~row 1: $|A| = a_{11} \text{ cof } a_{11} + a_{12} \text{ cof } a_{12} + a_{13} \text{ cof } a_{13}$~~

~~a_{11}~~

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};$$

row 1:

$$|A| = a_{11} \text{ cof } a_{11} + a_{12} \text{ cof } a_{12}$$
$$= a_{11} a_{22} - a_{12} a_{21}$$

row 2:

$$|A| = a_{21} \text{ cof } a_{21} + a_{22} \text{ Cof } a_{22}$$
$$= -a_{21} a_{12} + a_{22} a_{11}$$

column 2:

$$|A| = a_{12} \text{ cof } a_{12} + a_{22} \text{ cof } a_{22}$$
$$= -a_{12} a_{21} + a_{22} a_{11}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

~~column~~

row 2:

$$|A| = a_{21} \text{ cof } a_{21} + a_{22} \text{ cof } a_{22} + a_{23} \text{ cof } a_{23}$$
$$= -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= \overset{②}{\overline{a_{21} a_{13} a_{32}}} - \overset{①}{\overline{a_{21} a_{12} a_{33}}} + \overset{③}{\overline{a_{22} a_{11} a_{33}}} - \underset{④}{\underline{a_{22} a_{13} a_{31}}} + \underset{⑤}{\underline{a_{23} a_{31} a_{12}}} - \underset{⑥}{\underline{a_{23} a_{11} a_{32}}}$$

column 3:

$$|A| = a_{13} \text{ cof } a_{13} + a_{23} \text{ cof } a_{23} + a_{33} \text{ cof } a_{33}$$
$$= a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$
$$= \overset{②}{\overline{a_{13} a_{21} a_{32}}} - \overset{④}{\overline{a_{13} a_{22} a_{31}}} + \overset{⑤}{\overline{a_{23} a_{31} a_{12}}} - \underset{⑥}{\underline{a_{23} a_{11} a_{32}}} + \underset{③}{\underline{a_{33} a_{11} a_{22}}} - \underset{①}{\underline{a_{33} a_{21} a_{12}}}$$

Rule of Sarrus

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{21} & a_{22} \end{vmatrix}$$

Does <u>NOT</u> work for 2x2 matrices !!

~~Adjoint of a matrix~~ Cofactor matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\text{cof. matrix } A = \begin{pmatrix} \text{cof } a_{11} & \text{cof } a_{12} & \text{cof } a_{13} \\ \text{cof } a_{21} & \text{cof } a_{22} & \text{cof } a_{23} \\ \text{cof } a_{31} & \text{cof } a_{32} & \text{cof } a_{33} \end{pmatrix} = \text{cof}(A)$$

Adjoint of a matrix (Sometimes called adjugate)

$$\text{adj}(A) = (\text{cof}(A))^T = \tilde{A}$$

Inverse of a matrix

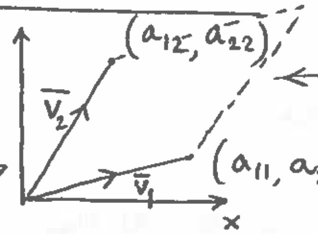
$$A^{-1} = \frac{\tilde{A}}{|A|} \; ; \quad \text{has the property} \quad A^{-1}A = AA^{-1} = I$$

If in some cases $|A| = 0$ then A is called a <u>singular matrix</u>. <u>Singular matrices cannot be inverted</u>.

What does a determinant actually mean
(geometric interpretation)

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \end{matrix}} \end{pmatrix}$$
$\qquad \bar{v}_1 \qquad \bar{v}_2$

$\mathbb{R}^2$

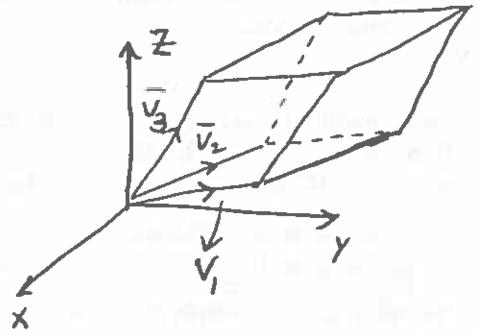


Oriented area of the parallelogram

oriented area is positive if $\bar{v}_1 \to \bar{v}_2$ is counterclockwise
" " " negative if $\bar{v}_1 \to \bar{v}_2$ " clockwise

$$\det(\bar{v}_1, \bar{v}_2) = -\det(\bar{v}_2, \bar{v}_1)$$

$$A = \begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}} & \boxed{\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}} \end{pmatrix}$$
$\qquad \bar{v}_1 \qquad \bar{v}_2 \qquad \bar{v}_3$



oriented ~~area~~ volume of the parallelopiped

Oriented volume is positive if $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ forms a rh system
" " " negative " " " " lh system.

If any two of the vectors are linearly dependent, then the area or volume $= 0 \Rightarrow \det A = 0$

<u>Matrix operations on vectors</u>

$$\underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{md} \end{pmatrix}}_{m \times d} \underbrace{\begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}}_{d \times 1} = \underbrace{\begin{pmatrix} a_{11} v_1 + \cdots + a_{1d} v_d \\ \cdots + \cdots + \cdots \\ a_{m1} v_1 + \cdots + a_{md} v_d \end{pmatrix}}_{m \times 1}$$

$m \times 1$ m-dimensional vector

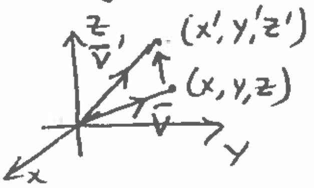
$$\underbrace{\begin{pmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}} & \boxed{\begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix}} & \boxed{\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix}} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} x + a_{12} y + a_{13} z \\ a_{21} x + a_{22} y + a_{23} z \\ a_{31} x + a_{32} y + a_{33} z \end{pmatrix}$$
$\qquad \bar{v}_1 \qquad \bar{v}_2 \qquad \bar{v}_3$

$$= x\bar{v}_1 + y\bar{v}_2 + z\bar{v}_3$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \bar{v}' = A\bar{v}$$

scale $\bar{v}_1$ by x and so on.

Transformation of the vector and <u>therefore point</u>



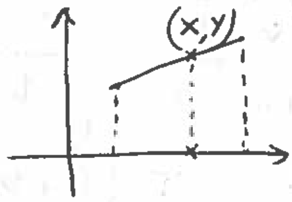
Projection

$$\begin{pmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & 0 \\ & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x\bar{v}_1 \quad \text{projection on to } x\text{-axis}$$

$\bar{v}_1$

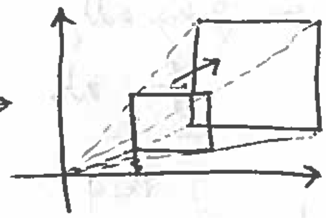
$$\begin{pmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0\\0\\0\end{matrix} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\bar{v}_1$$

$\bar{v}_1$



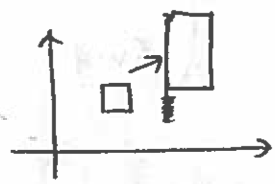
uniform scaling in 2D

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow$$



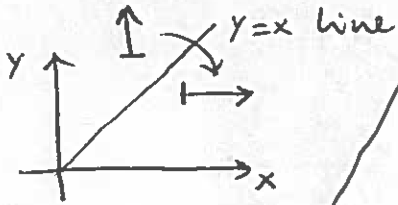
non-uniform scaling

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix} \longrightarrow$$



Reflection about $y = x$ line

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \leftarrow \text{swap } x \text{ and } y$$



Vector along $y = x$ line

$$\hat{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$a_{11} + a_{12} = 1$$
$$a_{21} + a_{22} = 1$$

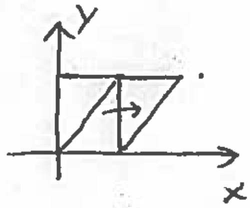
Vectors normal to $y = x$ line $\hat{u}_+ = \begin{pmatrix} -1 \\ +1 \end{pmatrix}$, $\hat{u}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$-a_{11} + a_{12} = 1$$
$$-a_{21} + a_{22} = -1 \Rightarrow a_{21} - a_{22} = 1$$

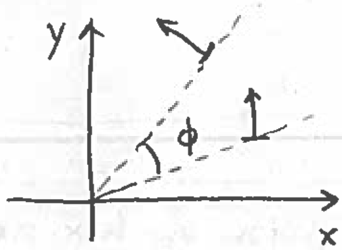
$$a_{11} = a_{22} = 0$$
$$a_{21} = a_{12} = 1$$

Shearing

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$



Rotation



anticlockwise rotation of object

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\parallel$

clockwise rotation of co-ordinate axes!

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \cdots\cdots$$

rotation along a ~~uniform~~ circle

> These belong to the so-called linear transformation of vectors
>
> i,e, $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$
>
> $T(c\bar{v}) = cT(\bar{v})$
>
> or generally $T(c_1\bar{u} + c_2\bar{v}) = c_1T(\bar{u}) + c_2T(\bar{v})$
>
> for all scalars c_1 and c_2.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad x'^2 + y'^2 = x^2 + y^2$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1}\underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$$

$$= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} \frac{\cos\phi}{a} & -\frac{\sin\phi}{b} \\ \frac{\sin\phi}{a} & \frac{\cos\phi}{b} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\frac{a}{b}\sin\phi \\ \frac{b}{a}\sin\phi & \cos\phi \end{pmatrix}$$