TUTORIAL 1 - SOLUTIONS
VECTORS, COORDINATE SYSTEMS AND PROJECTIONS IN 2D

PART 1: THEORY

EXERCISE 1

We start with some theoretical questions.

(a) Explain in your own words the difference between a scalar, a point and a vector, using an example in daily life.
(b) What is a unit vector?
(c) What is a 'basis' for a coordinate system?
(d) What is the 'norm' of a vector?
(e) Write down Pythagoras' theorem and explain it in your own words.

Solutions:

(a) A scalar is just a number, e.g. '3'. A point is a set of numbers, defined on a coordinate system, e.g., 1m away from both the bottom and the left edges of the table. A vector is an entity that specifies a direction and magnitude, e.g. wind velocity (speed + direction) in De uithof.
(b) A vector with length 1.
(c) A basis is a set of vectors, in terms of which every vector in a coordinate system can be expressed as a linear combination.
(d) The length of that vector.
(e) Draw a triangle. Call the hypotenuse c, the base a and height b. Then $a^2 + b^2 = c^2$. In the context of vectors, the length of the vector $\vec{c}$ can be calculated using this theorem.

PART 2: TWO-DIMENSIONAL VECTORS, INNER PRODUCTS

EXERCISE 2

Given two vectors $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Use the regular 2D Cartesian coordinate system (the x, y-system you know from highschool).

(a) Draw $\vec{a}$ and $\vec{b}$.
(b) Draw $\vec{a} + \vec{b}$.
(c) Draw $\vec{a} - \vec{b}$.
(d) Draw $2(\vec{a} - \vec{b})$.

Solutions:

(a) *
(b) *
(c) *
(d) *

EXERCISE 3

Given the following vectors: $\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \vec{c} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}, \vec{d} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{e} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}, \vec{f} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \vec{g} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \vec{h} = \begin{bmatrix} 2 \\ 3 \\ 8 \\ 1 \\ 2 \end{bmatrix}, \vec{i} = \begin{bmatrix} 1 \\ 5 \\ 2 \\ 1 \\ 6 \end{bmatrix}$.

Calculate the following vector problems:

(a) $\vec{a} + \vec{b}$
(b) $\vec{g} - \vec{h}$
(c) $\vec{a} + \vec{b} - \vec{c}$
(d) $||\vec{a} - \vec{b}||^2$
(e) $(||\vec{h}||^5 - ||\vec{i}||^3)\,(\vec{e} - \vec{f})$

Solutions:

(a) $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$
(b) $\begin{bmatrix} 3 \\ 1 \\ -5 \\ 1 \\ -1 \end{bmatrix}$
(c) $\begin{bmatrix} -3 \\ -3 \end{bmatrix}$
(d) 2
(e) $(82^{5/2} - 67^{3/2}) \cdot \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

Exercise 4

Given: two vectors in $\mathbb{R}^2$: $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\4\end{bmatrix}$ and $\vec{c} = \begin{bmatrix}5\\6\end{bmatrix}$. Calculate the following:

(a) $\vec{a}+\vec{b}$
(b) $\vec{b}-\vec{c}$
(c) $5\vec{b}$
(d) $||\vec{c}||^2$
(e) $||\vec{a}||$

Now in general form, set $\vec{a} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$, $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ and $\vec{c} = \begin{bmatrix}c_1\\c_2\end{bmatrix}$. Redo exercises (a)-(e).

Solutions:

(a) $\begin{bmatrix}4\\6\end{bmatrix}$
(b) $\begin{bmatrix}-2\\-2\end{bmatrix}$
(c) $\begin{bmatrix}15\\20\end{bmatrix}$
(d) 61
(e) $\sqrt{5}$

General forms:

(a) $\begin{bmatrix}a_1+b_1\\a_2+b_2\end{bmatrix}$
(b) $\begin{bmatrix}b_1-c_1\\b_2-c_2\end{bmatrix}$
(c) $\begin{bmatrix}5b_1\\5b_2\end{bmatrix}$
(d) $c_1^2+c_2^2$
(e) $\sqrt{a_1^2+a_2^2}$

Exercise 5

Given: two vectors in $\mathbb{R}^2$: $\vec{a} = \begin{bmatrix}3\\1\end{bmatrix}$, $\vec{b} = \begin{bmatrix}-1\\3\end{bmatrix}$ and $\vec{c} = \begin{bmatrix}-3\\-1\end{bmatrix}$. Calculate the following, and explain your answer geometrically.

(a) The length of $\vec{a}, \vec{b}$ and $\vec{c}$. For each of these calculations, draw an xy-plane and draw the triangles you implicitly use with Pythagoras' theorem.
(b) $\vec{a}+\vec{c}$
(c) Normalize $\vec{a}, \vec{b}$ and $\vec{c}$.

Solutions:

(a) For the calculation of $||\vec{a}||$, you implicitly use $||\vec{a}|| = \sqrt{a_x^2+a_y^2}$, so use the triangles that can be made by drawing the x and y coordinates of the points. This gives $||\vec{a}|| = \sqrt{10}$, $||\vec{b}|| = \sqrt{10}$ and $||\vec{c}|| = \sqrt{10}$.
(b) $\begin{bmatrix}0\\0\end{bmatrix}$. This cannot be drawn because it does not have a length that is larger than 0.
(c) $\hat{a} = \frac{1}{||\vec{a}||}\vec{a} = \begin{bmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}$, $\hat{b} = \frac{1}{||\vec{b}||}\vec{b} = \begin{bmatrix}-1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$, $\hat{c} = \frac{1}{||\vec{c}||}\vec{c} = \begin{bmatrix}-3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}$

Part 3: Norms, unit vectors and angles

Exercise 6

Draw the following vectors, then calculate the norm of each of them and write down the corresponding unit vectors:

(a) $\begin{bmatrix}2\\3\end{bmatrix}$
(b) $\begin{bmatrix}7\\9\end{bmatrix}$
(c) $\begin{bmatrix}\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\end{bmatrix}$
(d) $\begin{bmatrix}1\\2\\3\end{bmatrix}$
(e) $\begin{bmatrix}3\\1\\7\end{bmatrix}$
(f) $\begin{bmatrix}0\\0\\0\end{bmatrix}$

Solutions:

(a) $||\vec{a}|| = \sqrt{13}$, $\hat{a} = \begin{bmatrix}2/\sqrt{13}\\3/\sqrt{13}\end{bmatrix}$
(b) $||\vec{b}|| = \sqrt{130}$, $\hat{b} = \begin{bmatrix}7/\sqrt{130}\\9/\sqrt{130}\end{bmatrix}$
(c) $||\vec{c}|| = \sqrt{1}$, $\hat{c} = \begin{bmatrix}1/2\sqrt{2}\\1/2\sqrt{2}\end{bmatrix}$
(d) $||\vec{d}|| = \sqrt{14}$, $\hat{d} = \begin{bmatrix}1/\sqrt{14}\\2/\sqrt{14}\\3/\sqrt{14}\end{bmatrix}$
(e) $||\vec{e}|| = \sqrt{59}$, $\hat{e} = \begin{bmatrix}3/\sqrt{59}\\1/\sqrt{59}\\7/\sqrt{59}\end{bmatrix}$
(f) $||\vec{f}|| = \sqrt{0}$, $\hat{f}$ undefined.

EXERCISE 7

Given: three vectors in $\mathbb{R}^2$: $\vec{a} = \begin{bmatrix}1\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{c} = \begin{bmatrix}0\\1\end{bmatrix}$.

(a) Draw the vectors on a sheet of paper (using a regular 2D-Cartesian x,y-grid)
(b) Compute the sum of $\vec{a}$ and $\vec{b}$. Draw the result.
(c) Scale vector $\vec{a}$ by the factors 1, 2, -1 and $\sqrt{2}$. Draw the results.
(d) Determine $\vec{b} - \vec{c}$ graphically, using the rule from the lecture: first, join the starting points of b and c, then draw an arrow from the tip $\vec{c}$ to the tip of $\vec{b}$, which gives you the result.
(e) Compute and draw $2(\vec{a} + \vec{c})$ and $2\vec{a} + 2\vec{c}$. Visualize how this gives the same result (this is called the distributive rule).
(f) Build a linear combination $\vec{v}$ of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. That is, $\vec{v} = \lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c}$.

Solutions:

(a) *
(b) $\vec{a} + \vec{b} = \begin{bmatrix}2\\5\end{bmatrix}$
(c) Scaling with a scalar λ means $\lambda \cdot \vec{a} = \begin{bmatrix}\lambda\\3\lambda\end{bmatrix}$. Drawing all these scaled vectors results in a series of parallel vectors.
(d) $\vec{b} - \vec{c} = \begin{bmatrix}1\\1\end{bmatrix}$. Your drawn vectors should point towards the upper right.
(e) Computing both gives $\begin{bmatrix}2\\8\end{bmatrix}$. They are equal.
(f) Take $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then $\vec{v} = \begin{bmatrix}2\\6\end{bmatrix}$. You can check this geometrically.