TUTORIAL 2 - EXERCISES

VECTOR ALGEBRA (CONTINUED), PRIMITIVES AND PROJECTIONS IN 2D

PART 1: THEORY

EXERCISE 1

(a) What does it mean if two vectors are 'orthogonal'? And 'orthonormal'?
(b) What is a 'orthogonal basis' for a coordinate system?
(c) For what angle between two vectors is the dot product of those vectors at its largest?
(d) What is the relation between the magnitude of a vector and the dot product of the vector with itself?
(e) What do you need to know about a line in order to write its parametric form?
(f) Given the direction $\hat{v}$ of a line, can you write its parametric form? If yes, write it. If not, what else do you need?
(g) Write down a general line in the slope-intercept form. What is the meaning of every term?
(h) Given a segment of length l and direction $\hat{v}$, can you determine the length t of its projection on the x-axis? If yes, write it. If not, what else do you need?

PART 2: VECTOR ALGEBRA

EXERCISE 2

If $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\vec{a} \cdot \vec{b} = 25$. What is the length of $\vec{b}$ in the following cases for the angle θ between $\vec{a}$ and $\vec{b}$:

(a) $\theta = 0°$
(b) $\theta = 85°$
(c) $\theta = -45°$

What happens if $\theta = 90°$? (Hint: make a drawing)

EXERCISE 3

Given: two unit vectors $\hat{a}$ and $\hat{b}$. What do we know about the angle between these vectors in the following cases?

(a) If $\hat{a} \cdot \hat{b} = 0$
(b) If $\hat{a} \cdot \hat{b} < 0$
(c) If $\hat{a} \cdot \hat{b} = 1$

EXERCISE 4

Given the following vectors: $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{e} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix}$, $\vec{f} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, $\vec{g} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, $\vec{h} = \begin{pmatrix} 2 \\ 3 \\ 8 \\ 1 \\ 2 \end{pmatrix}$, $\vec{i} = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 1 \\ 6 \end{pmatrix}$.

Calculate the following vector problems:

(a) $\vec{d} \cdot \vec{a}$
(b) $\vec{e} \cdot \vec{f}$
(c) $\vec{g} \cdot \vec{h}$
(d) $(\vec{a} + \vec{d}) \cdot \vec{c}$
(e) $(\vec{f} \cdot \vec{e})\,\vec{g}$
(f) $(\vec{a} \cdot \vec{c})\,(\vec{b} + \vec{d})$
(g) $(\vec{f} + \vec{e})\,(\vec{i} \cdot \vec{a})$
(h) $\vec{i}\,(\vec{b} \cdot \vec{c}) + \vec{h} - \vec{g}$
(i) $(\vec{a} \cdot \vec{a})\,(\vec{a} \cdot \vec{a})$
(j) $((\vec{h} - \vec{i}) + ||\vec{g}||^2\,\vec{g}) \cdot \vec{h}$

PART 3: BASES AND ORTHOGONALITY

EXERCISE 5

Given: two vectors $\vec{a}$ and $\vec{b}$.

(a) The vectors form a 2D basis. What can you say about the dot product $\vec{a} \cdot \vec{b}$?
(b) The vectors form an orthonormal basis. What can you say about the magnitude of $\vec{b}$?
(c) Vector $\vec{a} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{pmatrix}$. Assuming orthonormality, write down all possibilities for vector $\vec{b}$.
(d) Point $p = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What are the coordinates of p in the 2D coordinate system defined by $\vec{a}$ and $\vec{b}$? Specify which $\vec{b}$ you used.

EXERCISE 6

Given: two vectors $\vec{a} = \begin{pmatrix}1\\2\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}4\\-2\end{pmatrix}$. Call $\vec{c} = \vec{a} + \vec{b}$.

(a) Calculate the length of $\vec{a}$, and of $\vec{b}$, and add them together.
(b) Calculate $\vec{c}$ and its length. Compare this to your answer in question (a). What do you notice?
(c) Explain your findings in question (b) by making a drawing, and making use of Pythagoras theorem. Why can you use Pythagoras theorem here?

EXERCISE 7

Given: the 2D vectors $\vec{a} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\vec{b} = \begin{pmatrix}4\\-1\end{pmatrix}$.

(a) Why are these vectors not orthogonal?
(b) Rewrite $\vec{b} = \begin{pmatrix}4\alpha\\-1\end{pmatrix}$. For what value of α are these vectors orthogonal?

EXERCISE 8

In 2D, an often used orthonormal basis in Cartesian coordinates is $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$. Given a point $P = (2, 9)$ in this basis. Give an example of an orthonormal basis in 2D of which one of the vectors is $\begin{pmatrix}\frac{1}{3}\\\frac{2}{3}\sqrt{2}\end{pmatrix}$. What is P in this new coordinate system?

PART 4: LINES AND GEOMETRY IN 2D

EXERCISE 9

Given a line in $\mathbb{R}^2$ that goes through two points (-4,1) and (2,2):

(a) What is the length of this line segment?
(b) What is the parametric representation of this line?
(c) Determine the unit vector normal to the line.
(d) Write the slope-intercept form and implicit form for this line.

EXERCISE 10

A ray is shot along the vector $\begin{pmatrix}1/5\\3/5\end{pmatrix}$ from point $(-1, 3)$, yielding a line L:

(a) Find the equation of the line L in all forms (parametric, slope-intecept and implicit).
(b) What the unit vector normal to this line?
(c) Find the equation of the line perpendicular to L at $(2, 12)$.

EXERCISE 11

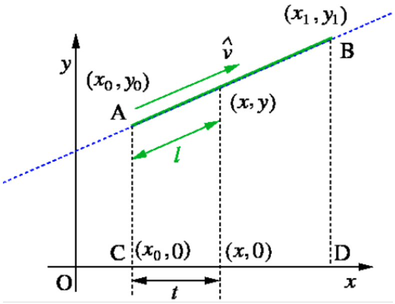
A line L goes through two points P_1 =(1,5) and P_2 =(3,7).

(a) Find the equation of the line L in all forms (parametric, slope-intecept and implicit).
(b) Find the equation of the line L', parallel to line L and passing through the origin, in all forms (parametric, slope-intecept and implicit).
(c) What is the length of the line segment between the two points (1,5) and (3,7) on line L?
(d) What is the length of the projection of the line segment on the x-axis? And on the y-axis?
(e) At point Q=(2,9) we put a spotlight shining towards the x-axis: what is the length of the shadow of the line segment between (1,5) and (3,7) on the x-axis?

EXERCISE 12

Given a set of points A, B, C and D as shown in the figure below. A line k passes through A and B. The points C and D are projections of A and B on the x-axis.

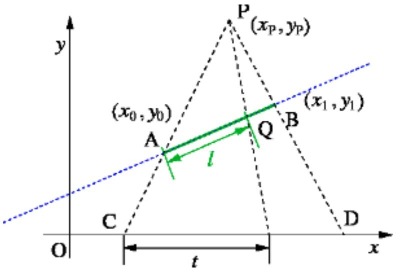
(a) Given that $A = (1, 2)$ and $B = (4, 6)$, give the coordinates of C and D.
(b) Find the equation of the line k in all forms (parametric, slope-intercept and implicit).
(c) Determine the normalized vector $\hat{v}$.
(d) Calculate a vector that is orthogonal to line k (hint: use your answer in (c))
(e) Give t as a function of l.



EXERCISE 13

Given a set of points A, B, C, D, Q and P as shown in the figure below (at P there is a light source, and the shadows of A and B on the x-axis are C and D respectively). A line k passes through A, Q and B.

(a) Given that $A = (1,2)$ and $B = (4,3)$, give the parametric equation of line l.
(b) Moreover, if $l = 1$, calculate the coordinates of Q.
(c) Calculate the coordinates of C and D, given that $P = (3,6)$.
(d) Determine t as a function of l.
(e) Now say that we do not know the length of l and the coordinates of Q. What should t be for QB to be equal to 1?



EXERCISE 14

A line L goes through two points $P_1 = (1,8)$ and $P_2 = (4,4)$.

(a) Find the equation of the line L in all forms (parametric, slope-intercept and implicit).
(b) A camera is placed on $(0,-4)$ and a screen is placed on the x-axis. Write the coordinates of the projections of P_1 and P_2 as the camera projects them on the screen.