Tutorial 2 - Solutions

Vector algebra (continued), primitives and projections in 2D

Part 1: Theory

Exercise 1

(a) What does it mean if two vectors are 'orthogonal'? And 'orthonormal'?
(b) What is a 'orthogonal basis' for a coordinate system?
(c) For what angle between two vectors is the dot product of those vectors at its largest?
(d) What is the relation between the magnitude of a vector and the dot product of the vector with itself?
(e) What do you need to know about a line in order to write its parametric form?
(f) Given the direction $\hat{v}$ of a line, can you write its parametric form? If yes, write it. If not, what else do you need?
(g) Write down a general line in the slope-intercept form. What is the meaning of every term?
(h) Given a segment of length l and direction $\hat{v}$, can you determine the length t of its projection on the x-axis? If yes, write it. If not, what else do you need?

Solutions:

(a) Orthogonal: vectors are perpendicular. Orthonormal: vectors are also unit vectors.
(b) A basis where all basis vectors are perpendicular to each other.
(c) $0°$
(d) $||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}}$
(e) The direction of the line and the coordinates of one of its points, or the coordinates of two points on the line.
(f) You need one point on the line.
(g) $y = mx + q$. q is the intercept, the y-coordinate of the point where the line crosses the y-axis; m is the tangent of the angle between the line and the x-axis.
(h) Yes. $t = l v_x$.

Part 2: Vector algebra

Exercise 2

If $\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{a} \cdot \vec{b} = 25$. What is the length of $\vec{b}$ in the following cases for the angle θ between $\vec{a}$ and $\vec{b}$:

(a) $\theta = 0°$
(b) $\theta = 85°$
(c) $\theta = -45°$

What happens if $\theta = 90°$? (Hint: make a drawing)

Solutions:
Use the following: $\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta)$

(a) Fill in the mentioned equation. Remember that $\cos(0) = 1$:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos(\theta) \\ 25 &= \sqrt{2 \cdot 2 + 3 \cdot 3} \cdot ||\vec{b}|| \cdot 1 \\ 25 &= \sqrt{13} \cdot ||\vec{b}|| \\ ||\vec{b}|| &= \frac{25}{\sqrt{13}} \end{aligned}$$

(b) Analogously, $||\vec{b}|| \approx 79.56$
(c) Analogously, $||\vec{b}|| \approx 9.81$

Then $\vec{a}$ and $\vec{b}$ are perpendicular, meaning that $\vec{a} \cdot \vec{b} = 0 \neq 25$.

Exercise 3

Given: two unit vectors $\hat{a}$ and $\hat{b}$. What do we know about the angle between these vectors in the following cases?

(a) If $\hat{a} \cdot \hat{b} = 0$
(b) If $\hat{a} \cdot \hat{b} < 0$
(c) If $\hat{a} \cdot \hat{b} = 1$

Solutions:

(a) Angle is $90°$
(b) Angle is between $90°$ and $270°$
(c) Angle must be $0°$, as it is given that $\hat{a}$ and $\hat{b}$ are unit vectors.

Exercise 4

Given the following vectors: $\vec{a}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{b}=\begin{bmatrix}2\\3\end{bmatrix}$, $\vec{c}=\begin{bmatrix}6\\8\end{bmatrix}$, $\vec{d}=\begin{bmatrix}4\\1\end{bmatrix}$, $\vec{e}=\begin{bmatrix}1\\6\\2\end{bmatrix}$, $\vec{f}=\begin{bmatrix}0\\2\\4\end{bmatrix}$, $\vec{g}=\begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$, $\vec{h}=\begin{bmatrix}2\\3\\8\\1\\2\end{bmatrix}$, $\vec{i}=\begin{bmatrix}1\\5\\2\\1\\6\end{bmatrix}$.

Calculate the following vector problems:

(a) $\vec{d}\cdot\vec{a}$
(b) $\vec{e}\cdot\vec{f}$
(c) $\vec{g}\cdot\vec{h}$
(d) $(\vec{a}+\vec{d})\cdot\vec{c}$
(e) $(\vec{f}\cdot\vec{e})\,\vec{g}$
(f) $(\vec{a}\cdot\vec{c})\,(\vec{b}+\vec{d})$
(g) $(\vec{f}+\vec{e})\,(\vec{b}\cdot\vec{a})$
(h) $\vec{i}\,(\vec{b}\cdot\vec{c})+\vec{h}-\vec{g}$
(i) $(\vec{a}\cdot\vec{a})\,(\vec{a}\cdot\vec{a})$
(j) $((\vec{h}-\vec{i})+||\vec{g}||^2\,\vec{g})\cdot\vec{h}$

Solutions:

(a) 6
(b) 20
(c) 50
(d) 54
(e) $\begin{bmatrix}100\\80\\60\\40\\20\end{bmatrix}$
(f) $\begin{bmatrix}132\\88\end{bmatrix}$
(g) $\begin{bmatrix}8\\64\\48\end{bmatrix}$
(h) $\begin{bmatrix}33\\179\\77\\35\\217\end{bmatrix}$
(i) 25
(j) 2786

Part 3: Bases and Orthogonality

Exercise 5

Given: two vectors $\vec{a}$ and $\vec{b}$.

(a) The vectors form a 2D basis. What can you say about the dot product $\vec{a}\cdot\vec{b}$?
(b) The vectors form an orthonormal basis. What can you say about the magnitude of $\vec{b}$?
(c) Vector $\vec{a}=\begin{bmatrix}\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\end{bmatrix}$. Assuming orthonormality, write down all possibilities for vector $\vec{b}$.
(d) Point $p=(1,2)$. What are the coordinates of p in the 2D coordinate system defined by $\vec{a}$ and $\vec{b}$? Specify which $\vec{b}$ you used.

Solutions:

(a) If they form a 2D basis, they are not parallel. This means that $\vec{a}\cdot\vec{b}\neq||\vec{a}||\,||\vec{b}||$.
(b) $||\vec{b}||=1$, because orthonormal vectors are also unit vectors.
(c) Orthonormality gives that $||\vec{b}||=1$, so $b_y=\pm\sqrt{1-b_x^2}$. Furthermore, $\vec{a}\cdot\vec{b}=0$, which results in the equation:

$$\frac{1}{2}\sqrt{2}b_x+\frac{1}{2}\sqrt{2}b_y=0 \tag{1}$$

Filling in $b_y=+\sqrt{1-b_x^2}$ into the above equation results in $b_x=\pm\sqrt{\frac{1}{2}}$. Filling in b_x into the same equation results in an equation with only b_y. Solving this results in $b_y=\pm\sqrt{\frac{1}{2}}$. It is important to now try out the possible values belong to each other, because while squaring the equation for solving b_x, you automatically lose the minus sign. So $\vec{b}=\begin{bmatrix}-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}\end{bmatrix}$ and $\begin{bmatrix}\frac{1}{2}\sqrt{2}\\-\frac{1}{2}\sqrt{2}\end{bmatrix}$ are the options.

(d) Use $\vec{b}=\begin{bmatrix}\frac{1}{2}\sqrt{2}\\-\frac{1}{2}\sqrt{2}\end{bmatrix}$. For these kind of questions, always make a drawing, because that will give you intuition on whether you are doing the right thing. Call $p_{ab}=(p_1,p_2)$ the point P in the basis with basisvectors $\vec{a},\vec{b}$. Set $p_1\vec{a}+p_2\vec{b}=\begin{bmatrix}1\\2\end{bmatrix}$.

$$p_1\frac{1}{2}\sqrt{2}+p_2\frac{1}{2}\sqrt{2}=1 \tag{2}$$
$$p_1\frac{1}{2}\sqrt{2}-p_2\frac{1}{2}\sqrt{2}=2 \tag{3}$$

This gives $p_1=3/\sqrt{2}$, $p_2=-1/\sqrt{2}$.

Exercise 6

Given: two vectors $\vec{a}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}4\\-2\end{bmatrix}$. Call $\vec{c}=\vec{a}+\vec{b}$.

(a) Calculate the length of $\vec{a}$, and of $\vec{b}$, and add them together.
(b) Calculate $\vec{c}$ and its length. Compare this to your answer in question (a). What do you notice?

(c) Explain your findings in question (b) by making a drawing, and making use of Pythagoras theorem. Why can you use Pythagoras theorem here?

Solutions:

(a) $||\vec{a}|| = \sqrt{5}$, $||\vec{b}|| = \sqrt{20}$. Adding them up gives $3\sqrt{5}$.
(b) $||\vec{c}|| = 5$. This is smaller than the answer in (a).
(c) We can use Pythagoras here because $\vec{a}$ and $\vec{b}$ are perpendicular (check by taking the dot product; which turns out to be zero). So we get $||\vec{c}|| = \sqrt{||\vec{a}||^2 + ||\vec{b}||^2} < ||\vec{a}|| + ||\vec{b}||$.

EXERCISE 7

Given: the 2D vectors $\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

(a) Why are these vectors not orthogonal?
(b) Rewrite $\vec{b} = \begin{bmatrix} 4\alpha \\ -1 \end{bmatrix}$. For what value of α are these vectors orthogonal?

Solutions:

(a) Because $\vec{a} \cdot \vec{b} = 5 \neq 0$.
(b) Determine the solution to the equation: $\vec{a} \cdot \vec{b} = 0$ with the new $\vec{b}$. This results in $\alpha = \frac{3}{8}$.

EXERCISE 8

In 2D, an often used orthonormal basis in Cartesian coordinates is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Given a point $P = (2, 9)$ in this basis. Give an example of an orthonormal basis in 2D of which one of the vectors is $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3}\sqrt{2} \end{bmatrix}$. What is P in this new coordinate system?

Solutions:

First we need to calculate the second basisvector. Call the given vector $\vec{a}$. Now we calculate $\vec{b}$. Orthonormality gives that $||\vec{b}|| = 1$, so $b_y = \pm\sqrt{1 - b_x^2}$. Furthermore, $\vec{a} \cdot \vec{b} = 0$, which results in the equation:

$$\frac{1}{3} b_x + \frac{2}{3}\sqrt{2} b_y = 0 \tag{4}$$

Filling in $b_y = +\sqrt{1 - b_x^2}$ into the above equation results in $b_x = \pm\sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$. If we now fill in these values of b_x into the same equation above, we can solve for b_y, which gives $b_y = \pm\sqrt{\frac{1}{9}} = \pm\frac{1}{3}$. This results in $\vec{b} = \begin{bmatrix} \frac{2}{3}\sqrt{2} \\ -\frac{1}{3} \end{bmatrix}$ is such a vector. Now calculate $P = (p_1, p_2)$ in this new basis. Set $p_1 \vec{a} + p_2 \vec{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

$$p_1 \frac{1}{3} + p_2 \frac{2}{3}\sqrt{2} = 2, \; p_1 \frac{2}{3}\sqrt{2} - p_2 \frac{1}{3} = 9. \tag{5}$$

This ultimately gives $p_1 = \frac{2(9\sqrt{2}+1)}{3}$ and $p_2 = \frac{4\sqrt{2}-9}{3}$.

PART 4: LINES AND GEOMETRY IN 2D

EXERCISE 9

Given a line in $\mathbb{R}^2$ that goes through two points (-4,1) and (2,2):

(a) What is the length of this line segment?
(b) What is the parametric representation of this line?
(c) Determine the unit vector normal to the line.
(d) Write the slope-intercept form and implicit form for this line.

Solutions:

(a) $\sqrt{(-4-2)^2 + (1-2)^2} = \sqrt{36+1} = \sqrt{37}$

(b)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + l \begin{bmatrix} 2+4 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + l \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

or, normalizing the direction of the line:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + \frac{t}{\sqrt{37}} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

(c) From (b) we can see that the direction of the line is $\frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$. Remembering that two vectors $\vec{v}$ and $\vec{w}$ are orthogonal to each other if and only if $\vec{v}\cdot\vec{w}=0$:

$$\begin{bmatrix}6\\1\end{bmatrix}\cdot\begin{bmatrix}v_x\\v_y\end{bmatrix}=6v_x+v_y=0.$$

A simple solution is $\begin{bmatrix}1/6\\-1\end{bmatrix}$ so, after normalization, the unit vector normal to the line can be written as $\pm\frac{1}{\sqrt{37}}\begin{bmatrix}1\\-6\end{bmatrix}$.

(d) Slope-intercept form:

$$y=\frac{1}{6}x+\left(1-\frac{1}{6}(-4)\right)=\frac{1}{6}x+1+\frac{2}{3}\ =\frac{1}{6}x+\frac{5}{3}$$

From that, the implicit form can be obtained:

$$x-6y+10=0.$$

Exercise 10

A ray is shot along the vector $\begin{bmatrix}1/5\\3/5\end{bmatrix}$ from point $(-1,3)$, yielding a line L:

(a) Find the equation of the line L in all forms (parametric, slope-intecept and implicit).
(b) What the unit vector normal to this line?
(c) Find the equation of the line perpendicular to L at $(2,12)$.

Solutions:

(a) Parametric is obtained from the given information:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}+l\begin{bmatrix}1/5\\3/5\end{bmatrix}$$

Slope-intercept:

$$y=\frac{3/5}{1/5}x+\left(3-\frac{3/5}{1/5}(-1)\right)=3x+6$$

Implicit:

$$3x-y+6=0$$

(b) We need the vector $\vec{w}$, satisfying

$$\begin{bmatrix}1/5\\3/5\end{bmatrix}\cdot\begin{bmatrix}w_x\\w_y\end{bmatrix}=0\Rightarrow 1/5w_x+3/5w_y=w_x+3w_y=0$$

A simple solution is $\begin{bmatrix}-1\\1/3\end{bmatrix}$ so, after normalization: $\pm\frac{1}{\sqrt{10/9}}\begin{bmatrix}-1\\1/3\end{bmatrix}$.

(c) Since we have all the information required (a point on the line and its direction) the parametric form is simply:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\12\end{bmatrix}+l\begin{bmatrix}-3\\1\end{bmatrix}$$

(there is no need to write the normalization term). The slope-intercept form can be derived from this:

$$y=-x/3+(12+2/3)=-\frac{x}{3}+\frac{38}{3}.$$

Exercise 11

A line L goes through two points $P_1=(1{,}5)$ and $P_2=(3{,}7)$.

(a) Find the equation of the line L in all forms (parametric, slope-intecept and implicit).
(b) Find the equation of the line L', parallel to line L and passing through the origin, in all forms (parametric, slope-intecept and implicit).
(c) What is the length of the line segment between the two points (1,5) and (3,7) on line L?
(d) What is the length of the projection of the line segment on the x-axis? And on the y-axis?
(e) At point Q=(2,9) we put a spotlight shining towards the x-axis: what is the length of the shadow of the line segment between (1,5) and (3,7) on the x-axis?

Solutions:

(a) $\vec{v}=\begin{bmatrix}3-1\\7-5\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$ or, normalized: $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$. Therefore the parametric form is: $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}+l\begin{bmatrix}1\\1\end{bmatrix}$.
Slope intercept: $y=x+(5-1)=x+4$.
Implicit: $x-y+4=0$.

(b) Take the slope intercept form and keep the directional coefficient (the '1' in front of x in $y=x+4$), while varying the constant ('4' in this case). This results in $L': y=x+c$, with c a constant. Now see for which c this line goes through the origin. This is true for $c=0$. So $y=x$. The implicit form follows to be $y-x=0$. The parametric form follows to be $\begin{bmatrix}x\\y\end{bmatrix}=l\begin{bmatrix}1\\1\end{bmatrix}$.

(c) $\sqrt{(3-1)^2+(7-5)^2} = \sqrt{8} = 2\sqrt{2}$.

(d) $v_x = \hat{v}\cdot\hat{x} = \frac{1}{\sqrt{2}}$, therefore the length of the projection on the x-axis is $t_x = l v_x = 2$.
$v_y = \hat{v}\cdot\hat{y} = \frac{1}{\sqrt{2}}$, therefore the length of the projection on the y-axis is $t_y = l v_y = 2$.

(e) The unit vectors along QP_1 and QP_2 are

$$\hat{w}_1 = \frac{P_1 - Q}{|P_1 - Q|} = -\frac{1}{\sqrt{17}}\begin{bmatrix}1\\4\end{bmatrix}$$
$$\hat{w}_2 = \frac{P_2 - Q}{|P_2 - Q|} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$$

In the slope-intercept form the equations of these two lines are:

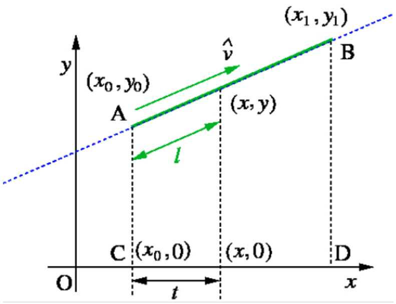
$$y = 4x+1 \quad \text{and} \quad y = -2x+13 \quad \text{respectively,}$$

intersecating with the x-axis at $(-1/4, 0)$ and $(13/2, 0)$ respectively, so the length of the segment on the x-axis is: $13/2 + 1/4 = 27/4$.

EXERCISE 12

Given a set of points A, B, C and D as shown in the figure below. A line k passes through A and B. The points C and D are projections of A and B on the x-axis.

(a) Given that $A = (1,2)$ and $B = (4,6)$, give the coordinates of C and D.
(b) Find the equation of the line k in all forms (parametric, slope-intercept and implicit).
(c) Determine the normalized vector $\hat{v}$.
(d) Calculate a vector that is orthogonal to line k (hint: use your answer in (c))
(e) Give t as a function of l.



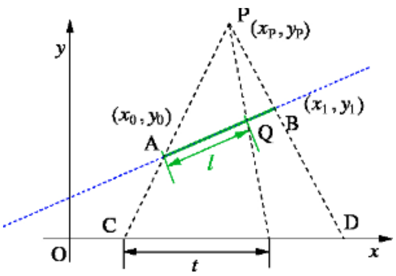
Solutions:

(a) $C = (1,0)$ and $D = (4,0)$.
(b) Slope-intercept form: $y = \frac{4}{3}x + \frac{2}{3}$
Implicit form: $\frac{4}{3}x + \frac{2}{3} - y = 0$, or $4x - 3y + 2 = 0$
Parametric form: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} + \frac{t}{5}\begin{bmatrix}3\\4\end{bmatrix}$ for varying t.
(c) $\vec{v}$ we actually already used in question (b), to be $\vec{v} = \begin{bmatrix}3\\4\end{bmatrix}$. Normalize this is $\hat{v} = \begin{bmatrix}3/5\\4/5\end{bmatrix}$.
(d) Use the fact that it would be orthogonal to vector $\vec{v}$ (dot product 0). One option for this is $\begin{bmatrix}4\\-3\end{bmatrix}$.
(e) $t = l(\hat{v}\cdot\hat{x}) = \frac{3}{5}l$.

EXERCISE 13

Given a set of points A, B, C, D, Q and P as shown in the figure below (at P there is a light source, and the shadows of A and B on the x-axis are C and D respectively). A line k passes through A, Q and B.

(a) Given that $A = (1,2)$ and $B = (4,3)$, give the parametric equation of the line k through A and B.
(b) Moreover, if $l = 1$, calculate the coordinates of Q.
(c) Calculate the coordinates of C and D, given that $P = (3,6)$.
(d) Determine t as a function of l.
(e) Now say that we do not know the length of l and the coordinates of Q. What should t be for QB to be equal to 1?



Solutions:

(a) $k : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{l}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ for varying l.

(b) Filling this into line k, we get $(x_Q, y_Q) = (1 + 3/\sqrt{10}, 2 + 1/\sqrt{10})$.

(c) Shoot a ray from P to A. In parametric form, we can write: $\begin{bmatrix} x_C \\ 0 \end{bmatrix} = \begin{bmatrix} x_P \\ y_P \end{bmatrix} + \alpha \begin{bmatrix} x_A - x_P \\ y_A - y_P \end{bmatrix}$, to solve for α and then x_C. This results in $0 = 6 - 4\alpha$, so that $\alpha = 3/2$, resulting in $C = (0,0)$. Doing the same for D results in $D = (5,0)$.

(d) First, calculate the coordinates of Q in terms of l. This gives $Q = (1 + 3l/\sqrt{10}, 2 + l/\sqrt{10})$. Then, obtain the shadow of Q on the x-axis (call this point E). Following the procedure from question (c) gives $\begin{bmatrix} x_E \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \gamma \begin{bmatrix} -2 + 3l/\sqrt{10} \\ -4 + l/\sqrt{10} \end{bmatrix}$, such that $\gamma = \frac{6}{4 - l/\sqrt{10}}$ and $x_E = (\frac{15l}{4\sqrt{10} - l}, 0)$.

(e) Note that $AB = \sqrt{10}$. So $QB = \sqrt{10} - l$. So if $QB = 1$, then $l = \sqrt{10} - 1$. We know that $t = x_E$ (the big expression in (d)). Filling in $l = \sqrt{10} - 1$ gives you what t needs to be.

EXERCISE 14

A line L goes through two points $P_1 = (1,8)$ and $P_2 = (4,4)$.

(a) Find the equation of the line L in all forms (parametric, slope-intercept and implicit).

(b) A camera is placed on $(0,-4)$ and a screen is placed on the x-axis. Write the coordinates of the projections of P_1 and P_2 as the camera projects them on the screen.

Solutions:

(a) Parametric:

$$\hat{v} = \frac{1}{|P_2 - P_1|} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + \frac{l}{5} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

Slope-intercept: $y = -\frac{4}{3}x + \frac{28}{3}$.

Implicit: $4x + 3y - 28 = 0$.

(b) The unit vectors along CP_1 and CP_2 are $\hat{w}_1$ and $\hat{w}_2$ respectively, where C is the location of the camera.

$$\hat{w}_1 = \frac{1}{\sqrt{145}} \begin{bmatrix} 1 \\ 12 \end{bmatrix} \quad \hat{w}_2 = \frac{1}{4\sqrt{5}} \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Let their projections on the screen be located at $(x'_1, 0)$ and $(x'_2, 0)$ respectively.

$$\begin{bmatrix} x'_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + t \begin{bmatrix} 1 \\ 12 \end{bmatrix} \quad \Rightarrow t = -2/3 \Rightarrow \quad x'_1 = 1/3$$

$$\begin{bmatrix} x'_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} + s \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad \Rightarrow s = -1/2 \Rightarrow \quad x'_2 = 2.$$