TUTORIAL 3 - EXERCISES

CIRCLES, ELLIPSES, LINES IN 3D, (HYPER)PLANES AND CROSS PRODUCT

Note: if you are running out of time, skip a few questions and give exercise 8 priority.

PART 1: THEORY

EXERCISE 1

We start with some theoretical questions.

(a) Explain in your own words the geometric interpretation of a cross product of two vectors (in 3D).
(b) Given a normal $\hat{n}$ of a plane, what is the general equation for the plane?
(c) Is the dot product a vector or a scalar? And what about the cross product?
(d) Write down the formula for the cross-product and for the inner product of vectors $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ in terms of their elements and the angle θ between them.
(e) Given vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, without any calculation, which direction do you think $\vec{c} = \vec{a} \times \vec{b}$ heads? Visualize this using the right-hand-rule (for example, say that the x direction is to your right, the y in front of you, and z upward). Now check which direction $\vec{d} = \vec{b} \times \vec{a}$ points to.
(f) Write the formula of a circle and of an ellipse. Show that the ellipse is a more general form of a circle.

PART 2: CIRCLES AND ELLIPSES

EXERCISE 2

Given a circle in $\mathbb{R}^2$ with radius $r = 4$ and center $c = (5, 1)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. with the circular co-ordinate system)
(c) Shooting a ray from the origin in the direction $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where does it intersect the circle?
(d) Represent your answer of (c) in the parametric co-ordinate for the circle.

EXERCISE 3

Given a circle C_1 in $\mathbb{R}^2$ with radius $r_1 = 5$ and center $c_2 = (4, 6)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the cycle?
(c) A ray is shot from (0,7) along the vector $\bar{v} = \begin{pmatrix} 1 \\ 1/3 \end{pmatrix}$. Determine the points P_1 and P_2 where the ray passes through the circle.
(d) What is the length of the segment P that connects P_1 and P_2?
(e) What is the length of the parallel projection of P on the x-axis?

EXERCISE 4

Given a circle C_2 in $\mathbb{R}^2$ with radius $r = 2$ and center $c = (3, 2)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. with the circular co-ordinate system)
(c) Find the points of intersection P_1 and P_2 between this circle and C_1, the circle of Exercise 5. This was not shown in class, but think about it!
(d) What is the length of the segment P that connects P_1 and P_2?
(e) What is the length of the projection of P on the y-axis?

EXERCISE 5

Given the equation

$$x^2 + y^2 - 4y - 5 = 0$$

(a) Is it a circle or an ellipse? Why?
(b) Write the corresponding implicit form.
(c) Where is the center? If it's a circle, what is the radius? If it's an ellipse, what are the lengths of its semi-axes?

Exercise 6

Given a circle C_3 with center $c = (1,3)$ and radius 2.

(a) Write down the implicit and parametric representations of the circle.
(b) What's the maximum radius that a circle C_4 centered on the origin can have without intersecating C_3?
(c) If C_4 (still centered on the origin) passes through the center of C_3, what is its radius? Write down its implicit representation.
(d) Compute the coordinates of the points of intersection between C_3 and C_4.

Part 3: Dot and cross products in 3D

Exercise 7

Given coordinates $A = (1,2,3)$ and $B = (4,5,6)$.

(a) Calculate the equation for the line l going through A and B.
(b) Obtain a unit normal vector $\hat{n}$ to this line. Check your answer by explicitly calculating the distance of line l to point B.
(c) Calculate the distance of line l to point $C = (7,8,9)$.

Exercise 8

Given are the following vectors: $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}3\\4\\5\end{bmatrix}$, $\vec{c} = \begin{bmatrix}1\\3\\1\end{bmatrix}$, $\vec{d} = \begin{bmatrix}0\\2\\2\end{bmatrix}$, $\vec{e} = \begin{bmatrix}5\\3\\1\end{bmatrix}$, $\vec{f} = \begin{bmatrix}4\\2\\1\end{bmatrix}$, $\vec{g} = \begin{bmatrix}6\\2\\4\end{bmatrix}$.
Calculate and draw the following vectors in a 3D Cartesian coordinate system:

(a) $\vec{a} + \vec{b}$
(b) $\vec{c} + \vec{d}$
(c) $\vec{f} - \vec{g}$
(d) $\vec{e} \times \vec{a}$
(e) $\vec{d} - \vec{e} + \vec{a} \times \vec{f}$
(f) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$
(g) $((\vec{g} \times \vec{a}) \cdot (\vec{d} \times \vec{e}))\vec{c}$
(h) $(\vec{a} \times \vec{a} + \vec{a} \times \vec{a}) \cdot ||\vec{a}||^2$
(i) $(\vec{b} \times \vec{c}) + \vec{d} - \vec{a}$
(j) $(5\vec{a} \times \vec{a}) \times \vec{a}$

Exercise 9

Given vectors $\vec{u} = \begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

(a) Draw these vectors.
(b) Calculate and draw the dot product of these vectors. What is the angle between vectors $\vec{u}$ and $\vec{v}$?
(c) Calculate and draw the cross product of these vectors. What is the angle between the newly attained vector and $\vec{u}$?

Part 4: Planes and normals in 3D

Exercise 10

Given coordinates $A = (0,2,0)$, $B = (1,3,4)$ and $C = (-2,-2,3)$.

(a) Determine the normalized vector going from A in the direction of B. Call this $\hat{u}$.
(b) Determine the normalized vector going from A in the direction of C. Call this $\hat{v}$.
(c) Determine a normal vector to the plane derived through A, B and C.
(d) Determine the equation of this plane in the implicit form.
(e) Determine the equation of this plane in the parametric form.