Tutorial 3 - Solutions
Vectors, primitives and projections in 3D

Part 1: Theory

Exercise 1

We start with some theoretical questions.

(a) Explain in your own words the geometric interpretation of a cross product of two vectors (in 3D).
(b) Given a normal $\hat{n}$ of a plane, what is the general equation for the plane?
(c) Is the dot product a vector or a scalar? And what about the cross product?
(d) Write down the formula for the cross-product and for the inner product of vectors $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ in terms of their elements and the angle θ between them.
(e) Given vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, without any calculation, which direction do you think $\vec{c} = \vec{a} \times \vec{b}$ heads? Visualize this using the right-hand-rule (for example, say that the x direction is to your right, the y in front of you, and z upward). Now check which direction $\vec{d} = \vec{b} \times \vec{a}$ points to.
(f) Write the formula of a circle and of an ellipse. Show that the ellipse is a more general form of a circle.

Solutions:

(a) the cross product $\vec{w}$ of two vectors $\vec{u}$ and $\vec{v}$ is a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$ and has magnitude $||\vec{w}|| = ||\vec{u}||\,||\vec{v}||\sin\theta$
(b) $(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0$, where (x_0, y_0, z_0) is a point on the plane.
(c) The dot product is indeed a scalar, while cross product is a vector
(d) $\vec{w} = \vec{u} \times \vec{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$ and $||\vec{w}|| = ||\vec{u}||\,||\vec{v}||\sin\theta$
(e) $\vec{c}$ points towards your back (negative y), $\vec{d}$ points towards your front (positive y).
(f) Circle: $(x - x_c)^2 + (y - y_c)^2 = r^2$, with (x_c, y_c) the center of the circle and r its radius.
ellipse $\frac{(x-x_c)^2}{a^2} + \frac{(y-y_c)^2}{b^2} = 1$, with (x_c, y_c) the center of the ellipse, a and b are the lengths of the two semi-axes.
If $a = b$:

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{a^2} = \frac{(x - x_c)^2 + (y - y_c)^2}{a^2} = 1$$
$$(x - x_c)^2 + (y - y_c)^2 = a^2$$

which is a circle of radius a.

Part 2: Circles and ellipses

Exercise 2

Given a circle in $\mathbb{R}^2$ with radius $r = 4$ and center $c = (5, 1)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. with the circular co-ordinate system)
(c) Shooting a ray from the origin in the direction $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where does it intersect the circle?
(d) Represent your answer of (c) in the parametric co-ordinate for the circle.

Solutions:

(a) $(x - 5)^2 + (y - 1)^2 = 16$
(b) $x = 5 + 4\cos\theta,\ y = 1 + 4\sin\theta$
(c) Writing the ray in a parametric form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + l \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x = l, \quad y = l$$

by placing this result inside the implicit representation of the circle we get:

$$(l - 5)^2 + (l - 1)^2 = 16 \Rightarrow 2l^2 - 12l + 10 = 0 \Rightarrow l^2 - 6l + 5 = 0$$

this equation has two solutions: $P_1 = (5, 5)$ and $P_2 = (1, 1)$.

(d) For P_1:

$$\begin{bmatrix}x_1\\y_1\end{bmatrix}=\begin{bmatrix}5+4\cos\theta_1\\1+4\sin\theta_1\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}\Rightarrow\begin{bmatrix}\cos\theta_1\\\sin\theta_1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow\theta_1=\pi/2$$

For P_2:

$$\begin{bmatrix}x_2\\y_2\end{bmatrix}=\begin{bmatrix}5+4\cos\theta_2\\1+4\sin\theta_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\Rightarrow\begin{bmatrix}\cos\theta_2\\\sin\theta_2\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}\Rightarrow\theta_2=\pi$$

EXERCISE 3

Given a circle C_1 in $\mathbb{R}^2$ with radius $r_1=5$ and center $c_2=(4,6)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the cycle?
(c) A ray is shot from (0,7) along the vector $\bar{v}=\begin{pmatrix}1\\1/3\end{pmatrix}$. Determine the points P_1 and P_2 where the ray passes through the circle.
(d) What is the length of the segment P that connects P_1 and P_2?
(e) What is the length of the parallel projection of P on the x-axis?

Solutions:

(a) $(x-4)^2+(y-6)^2=25$.

(b) $x=4+5\cos\theta,\ y=6+5\sin\theta$.

(c) We can write the ray as $y=1/3x+7$. We can insert this inside the implicit representation of C_1:

$$25=(x-4)^2+(x/3+7-6)^2=(x-4)^2+(1/3x+1)^2=\frac{10}{9}x^2-\frac{22}{3}x+17$$

$$0=\frac{10}{9}x^2-\frac{22}{3}x-8\ \Rightarrow\ x_{1,2}=\frac{33}{10}\pm\frac{3\sqrt{201}}{10}$$

and therefore:

$$P_1=\left(\frac{33}{10}+\frac{3\sqrt{201}}{10},\frac{81+\sqrt{201}}{10}\right)\quad P_2=\left(\frac{33}{10}-\frac{3\sqrt{201}}{10},\frac{81-\sqrt{201}}{10}\right).$$

(d)

$$l=|P_2-P_1|=\sqrt{\left(\frac{6\sqrt{201}}{10}\right)^2+\left(\frac{2\sqrt{201}}{10}\right)^2}=\sqrt{\frac{(36+4)\,201}{100}}=2\sqrt{\frac{2010}{100}}=\frac{\sqrt{2010}}{5}.$$

(e) The length of the projection t can be easily found from $\hat{v}$:

$$t=lv_x=\frac{l}{\sqrt{10/9}}=\frac{3}{5}\sqrt{201}.$$

EXERCISE 4

Given a circle C_2 in $\mathbb{R}^2$ with radius $r=2$ and center $c=(3,2)$:

(a) What is the implicit representation of the circle?
(b) What is the parametric representation of the circle? (i.e. with the circular co-ordinate system)
(c) Find the points of intersection P_1 and P_2 between this circle and C_1, the circle of Exercise 3. This was not shown in class, but think about it!
(d) What is the length of the segment P that connects P_1 and P_2?
(e) What is the length of the projection of P on the y-axis?

Solutions:

(a) $(x-3)^2+(y-2)^2=4$

(b) $x=3+2\cos\theta\quad y=2+2\sin\theta$

(c) Solving the system of equations:

$$\begin{cases}(x-3)^2+(y-2)^2=4\\(x-4)^2+(y-6)^2=25\end{cases}\ \Rightarrow\ 2x+8y=18\Rightarrow\ x=9-4y$$

We put this result in C_2 and we solve the second degree equation:

$$(9-4y-3)^2+(y-2)^2=4\ \Rightarrow\ 17y^2-52y+36=0\ \Rightarrow\ y_{1,2}=\begin{cases}2\\18/17\end{cases}.$$

The two points are $P_1=(1,2)$ and $P_2=(81/17,18/17)$.

(d)

$$l=\sqrt{\left(\frac{81}{17}-1\right)^2+\left(\frac{18}{17}-2\right)^2}=\sqrt{\frac{4096}{289}+\frac{256}{289}}=\sqrt{\frac{4352}{289}}=\frac{16}{\sqrt{17}}.$$

(e) We already have the equation of the line connecting the two points ($x+4y-9=0$), which can be rewriten in a parametric form by considering two points on it. We will use two simple points: $(0,9/4)$ and $(1,2)$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 9/4 \end{bmatrix} + t\hat{v} = \begin{bmatrix} 0 \\ 9/4 \end{bmatrix} + \frac{t}{\sqrt{17/16}} \begin{bmatrix} 1 \\ -1/4 \end{bmatrix}$$

And then:

$$l_y = |\hat{v} \cdot \hat{y}| l = \frac{\sqrt{16}}{4\sqrt{17}} \frac{16}{\sqrt{17}} = \frac{16}{17}.$$

Exercise 5

Given the equation

$$x^2 + y^2 - 4y - 5 = 0$$

(a) Is it a circle or an ellipse? Why?
(b) Write the corresponding implicit form.
(c) Where is the center? If it's a circle, what is the radius? If it's an ellipse, what are the lengths of its semi-axes?

Solutions:

(a) It is a circle; see below.

(b) Let us rearrange the terms and

$$x^2 + y^2 - 4y = 5$$

as

$$x^2 + (y-2)^2 = 9$$

(c) It is a circle with center $(0,2)$ and radius 3.

Exercise 6

Given a circle C_3 with center $c = (1,3)$ and radius 2.

(a) Write down the implicit and parametric representations of the circle.
(b) What's the maximum radius that a circle C_4 centered on the origin can have without intersecating C_3?
(c) If C_4 (still centered on the origin) passes through the center of C_3, what is its radius? Write down its implicit representation.
(d) Compute the coordinates of the points of intersection between C_3 and C_4.

Solutions:

(a) $(x-1)^2 + (y-3)^2 = 4$.
$x = 1 + 2\cos\theta$, $y = 3 + 2\sin\theta$.

(b) We could write down the equation of C_4 and compute the intersection, determining for which value of r we have a single point of intersection and take that as our maximus radius. Or we could be smart about it: the distance between the center of C_3 and the origin is $\sqrt{10}$, its radius is 2, so the distance between the origin and C_3 is $\sqrt{10}-2$: this will be our maximum radius for C_4.

(c) As we have seen before, $\sqrt{10}$. $x^2 + y^2 = 10$.

(d) Solving the system:

$$\begin{cases} x^2 + y^2 = 10 \\ (x-1)^2 + (y-3)^2 = 4 \end{cases} \Rightarrow x + 3y = 8$$

By placing this inside the equation of C_4 we get the two points of intersection: $(-1,3)$ and $(13/5, 9/5)$.

Part 3: Dot and cross products in 3D

Exercise 7

Given coordinates $A = (1,2,3)$ and $B = (4,5,6)$.

(a) Calculate the equation for the line l going through A and B.
(b) Obtain a unit normal vector $\hat{n}$ to this line. Check your answer by explicitly calculating the distance of line l to point B.
(c) Calculate the distance of line l to point $C = (7,8,9)$.

Solutions:

(a) Direction vector $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, so parametric form of line l: $\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3}$

(b) The unit normal vector is perpendicular to $\vec{v}$, for example $\hat{n} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$ (check that $\hat{n}\cdot\vec{v}=0$. Now shoot a ray from B to l and from A along l and see where they intersect:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x_b\\y_b\\z_b\end{bmatrix} + t\hat{n_b} \tag{1}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x_a\\y_a\\z_a\end{bmatrix} + k\hat{v} \tag{2}$$

where $\hat{n_b}$ is the normal vector from the line through B. Now we need to solve this for t and k. For arbitrary $\hat{n_b}$ it is already visible that this only has a solution for $t=0$, so distance between B and the line is indeed 0.

(c) We see that C already holds the line equation $\frac{x-1}{3} = \frac{y-2}{3} = \frac{z-3}{3}$, so C is on the line: distance 0.

Exercise 8

Given are the following vectors: $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \vec{b} = \begin{bmatrix}3\\4\\5\end{bmatrix}, \vec{c} = \begin{bmatrix}1\\3\\1\end{bmatrix}, \vec{d} = \begin{bmatrix}0\\2\\2\end{bmatrix}, \vec{e} = \begin{bmatrix}5\\3\\1\end{bmatrix}, \vec{f} = \begin{bmatrix}4\\2\\1\end{bmatrix}, \vec{g} = \begin{bmatrix}6\\2\\4\end{bmatrix}$.

Calculate and draw the following vectors in a 3D Cartesian coordinate system:

(a) $\vec{a}+\vec{b}$
(b) $\vec{c}+\vec{d}$
(c) $\vec{f}-\vec{g}$
(d) $\vec{e}\times\vec{a}$
(e) $\vec{d}-\vec{e}+\vec{a}\times\vec{f}$
(f) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$
(g) $((\vec{g}\times\vec{a})\cdot(\vec{d}\times\vec{e}))\vec{c}$
(h) $(\vec{a}\times\vec{a}+\vec{a}\times\vec{a})\cdot||\vec{a}||^2$
(i) $(\vec{b}\times\vec{c})+\vec{d}-\vec{a}$
(j) $(5\vec{a}\times\vec{a})\times\vec{a}$

Solutions:

(a) $\begin{bmatrix}4\\6\\8\end{bmatrix}$

(b) $\begin{bmatrix}1\\5\\3\end{bmatrix}$

(c) $\begin{bmatrix}-2\\0\\-3\end{bmatrix}$

(d) $\begin{bmatrix}e_y a_z - e_z a_y\\ e_z a_x - e_x a_z\\ e_x a_y - e_y a_x\end{bmatrix} = \begin{bmatrix}7\\-14\\7\end{bmatrix}$

(e) $\begin{bmatrix}-5\\-1\\1\end{bmatrix} + \begin{bmatrix}2-6\\12-1\\2-8\end{bmatrix} = \begin{bmatrix}-9\\10\\-5\end{bmatrix}$

(f) $\begin{bmatrix}-2\\4\\-2\end{bmatrix} \times \begin{bmatrix}4\\-2\\2\end{bmatrix} = \begin{bmatrix}4\\-4\\-12\end{bmatrix}$

(g) $\left(\begin{bmatrix}-2\\-14\\10\end{bmatrix}\cdot\begin{bmatrix}-4\\10\\-10\end{bmatrix}\right)\cdot\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}-232\\-696\\-232\end{bmatrix}$

(h) $(\vec{0}+\vec{0})\cdot 14 = \vec{0}$

(i) $\begin{bmatrix}-11\\2\\5\end{bmatrix} + \begin{bmatrix}-1\\0\\-1\end{bmatrix} = \begin{bmatrix}-12\\2\\4\end{bmatrix}$

(j) $\begin{bmatrix}0\\0\\0\end{bmatrix} \times \vec{a} = \vec{0}$

Exercise 9

Given vectors $\vec{u} = \begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

(a) Draw these vectors.
(b) Calculate and draw the dot product of these vectors. What is the angle between vectors $\vec{u}$ and $\vec{v}$?
(c) Calculate and draw the cross product of these vectors. What is the angle between the newly attained vector and $\vec{u}$?

Solutions:

(a) *
(b) $\vec{u}\cdot\vec{v} = 1+0+6 = 7$. This is a scalar and does not have a direction, unlike a vector. Therefore, we do not draw it here. You can, however, relate the magnitude of this scalar to the angle between $\vec{u}$ and $\vec{v}$ via $||\vec{w}|| = ||\vec{u}||\,||\vec{v}||\cos\theta$, meaning $\theta = \arccos(\frac{7}{\sqrt(5)\sqrt(11)}) \approx 19.3°$.
(c) $\vec{u}\times\vec{v} = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$. The angle between this vector and $\vec{u}$ is 90°, by definition (cross product is perpendicular to the two vectors). In your drawings, it should be visible that the cross product is perpendicular to $\vec{u}$ and $\vec{v}$.

Part 4: Planes and normals in 3D

Exercise 10

Given coordinates $A=(0,2,0)$, $B=(1,3,4)$ and $C=(-2,-2,3)$.

(a) Determine the normalized vector going from A in the direction of B. Call this $\hat{u}$.
(b) Determine the normalized vector going from A in the direction of C. Call this $\hat{v}$.

(c) Determine a normal vector to the plane derived through A, B and C.
(d) Determine the equation of this plane in the implicit form.
(e) Determine the equation of this plane in the parametric form.

Solutions:

(a) $\hat{u} = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

(b) $\hat{v} = \frac{1}{\sqrt{29}} \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$

(c) The cross product of $\hat{u} \times \hat{v}$ provides us with the normal to the sought plane. $\hat{u} \times \hat{v} = \frac{1}{\sqrt{522}} \begin{bmatrix} 19 \\ -11 \\ -2 \end{bmatrix}$.

(d) Then the implicit form of the plane is $19x - 11y - 2z + 22 = 0$ (obtained by filling in the coordinates of A).

(e) The parametric form is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \frac{\alpha}{\sqrt{18}} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + \frac{\beta}{\sqrt{29}} \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$.