TUTORIAL 4 - SOLUTIONS

PRIMITIVES (CONTINUED) AND PROJECTIONS IN 3D

PART 1: THEORY

EXERCISE 1

(a) Give the general implicit formula of a sphere.
(b) Give the general parametric formula of a sphere.
(c) Given the implicit equation for a plane in 3D, give a general formulation of a normalized normal vector to this plane.

Solutions:

(a) Implicit sphere formula: $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$ with (x_0,y_0,z_0) the origin and radius r.

(b) Parametric form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + r\cdot\hat{u}_r$ with $\hat{u}_r = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$

(c) Given that the normalized normal vector is $\vec{v} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$, then the implicit equation for a plane in 3D is $Ax+By+Cz+D=0$, where D is a constant.

PART 2: PLANES AND PROJECTIONS IN 3D

EXERCISE 2

Given coordinates $A=(-1,1,0)$, $B=(1,-3,1)$ and $C=(-2,-2,-2)$. (See exercise 10 in tutorial 3 as a reference.)

(a) Determine the equation of the plane through A, B and C (parametric form).
(b) Determine the equation of this plane in the implicit form.

Solutions:

(a) Determine the unit vector from A to B: $\hat{v} = \frac{1}{\sqrt{21}}\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, and also the one from A to C: $\hat{u} = \frac{1}{\sqrt{14}}\begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$. Then the parametric form is, starting from A: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{\alpha}{\sqrt{21}}\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} + \frac{\beta}{\sqrt{14}}\begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}$.

(b) Use the cross product of $\hat{v}$ and $\hat{u}$; $\hat{n} = \hat{u}\times\hat{v} = -\frac{1}{\sqrt{294}}\begin{bmatrix} 11 \\ 3 \\ -10 \end{bmatrix}$. So the implicit equation is $11x+3y-10z+8=0$ (gained by filling in the coordinates of any of the points).

EXERCISE 3

Given a plane that is given by $3x+1y+6z-2=0$. Calculate the distance of the following points to this plane:

(a) $A=(0,0,0)$
(b) $B=(1,1,1)$
(c) $C=(2,-5,3)$

Solutions:

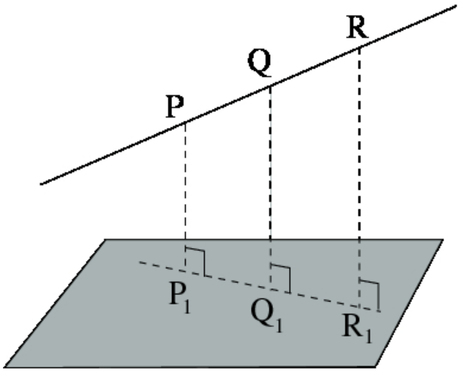
Shoot a ray $\vec{v}$ from the points towards the plane, and solve for the distance $|a|$: $3(x_0+av_x)+(y_0+av_y)+6(z_0+av_z)-2=0$. The vector $\vec{v}$ should be perpendicular to the plane, so this is found from the implicit planar equation: $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$, normalized we get $\hat{v} = \begin{bmatrix} 3/\sqrt{46} \\ 1/\sqrt{46} \\ 6/\sqrt{46} \end{bmatrix}$.

(a) Solve $3(0+3a/\sqrt{46})+(0+a/\sqrt{46})+6(0+6a/\sqrt{46})-2=0$. This results in $9a+a+36a-2\sqrt{46}=0$, so $a=2\sqrt{46}/46=\sqrt{46}/23=|a|$.
(b) Solve $3(1+3a/\sqrt{46})+(1+a/\sqrt{46})+6(1+6a/\sqrt{46})-2=0$. This results in $3+9a/\sqrt{46}+1+a/\sqrt{46}+6+36a/\sqrt{46}-2=0$, so $46a=-8\sqrt{46}$, or $a=-4\sqrt{46}/23$; $|a|=4\sqrt{46}/23$.
(c) Solve $3(2+3a/\sqrt{46})+(-5+a/\sqrt{46})+6(3+6a/\sqrt{46})-2=0$. This results in $6+9a/\sqrt{46}-5+a/\sqrt{46}+18+36a/\sqrt{46}-2=0$, so $a=-17\sqrt{46}/46$; $|a|=17\sqrt{46}/46$

Exercise 4

Given two points $P = (3,4,5)$ and $Q = (5,8,9)$.

(a) Calculate the line l going through P and Q in parametric form.
(b) Then project the piece of line between P and Q on the xy-plane (see figure; their projections are P_1 and Q_1). Calculate the length of this projected piece of line.
(c) Now consider another point R on line l. Say that $QR = t$, calculate the length of Q_1R_1 in terms of t.



Solutions:

(a) Unit vector $\hat{v} = (Q-P)/|Q-P| = \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}$. Then the equation of the line in parametric form is $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\4\\5\end{bmatrix} + k\cdot\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$.
(b) The coordinates of P_1 and Q_1 are relatively simple because they are projected perpendicular on the xy-plane. This means that $P_1 = (3,4,0)$ and $Q_1 = (5,8,0)$. The distance between P_1 and Q_1 is then $d = \sqrt{(5-3)^2+(8-4)^2} = \sqrt{20}$.
(c) The coordinates of R are $R = Q + t\cdot\vec{v} = (5+t/3, 8+2t/3, 9+2t/3)$. The projection of this point on the xy-plane is $(5+t/3, 8+2t/3, 0)$. Then the length Q_1R_1 equals $\sqrt{(t/3)^2+(2t/3)^2} = t\sqrt{\frac{5}{9}}$.

Exercise 5

Given two points $P = (3,4,5)$ and $R = (5,8,9)$, and camera at point $E = (4,4,-5)$. The xy-plane is the screen.

(a) Project PR to P_1R_1 on the screen as seen by the camera (see figure). Obtain the coordinates of P_1 and R_1.
(b) Given $PQ = l$, calculate the coordinates of point Q_1 in the xy-plane.

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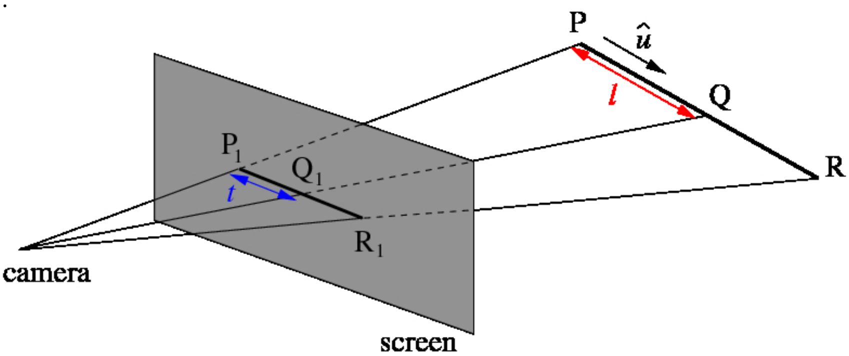

Solutions:

(a) Unit vector from P to E is $\hat{d_p} = (E-P)/|E-P| = \frac{1}{\sqrt{101}}\begin{bmatrix}1\\0\\-10\end{bmatrix}$. Now solve $\begin{bmatrix}x_P\\y_P\\z_P\end{bmatrix} + k\hat{d_p} = \begin{bmatrix}x_{p1}\\y_{p1}\\0\end{bmatrix}$. This gives:

$$\begin{cases} 3+k\frac{1}{\sqrt{101}} & = x_{p1} \\ 4 & = y_{p1} \\ 5-k\cdot\frac{10}{\sqrt{101}} & = 0 \end{cases} \tag{1}$$

From the third equation, we get $k = \frac{1}{2}\sqrt{101}$. The second equation directly gives $y_{p1} = 4$, and filling the attained k value into the first equation gives x_{p1}. So $P_1 = (\frac{7}{2}, 4, 0)$. We do the exact same thing for R and R_1, which results in a unit vector of $\hat{d_r} = -\frac{1}{\sqrt{213}}\begin{bmatrix}1\\4\\14\end{bmatrix}$. Doing the same as above, we get:

$$\begin{cases} 5-k\cdot\frac{1}{\sqrt{213}} & = x_{r1} \\ 8-k\cdot\frac{4}{\sqrt{213}} & = y_{r1} \\ 9-k\cdot\frac{14}{\sqrt{213}} & = 0 \end{cases} \tag{2}$$

which results in $R_1 = (\frac{61}{14}, \frac{38}{7}, 0)$.

(b) For this, first determine the coordinates of Q. Unit vector from P to R is $\hat{u} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. So $Q = (3 + l/3, 4 + 2l/3, 5 + 2l/3)$. Projecting this point on the screen, analogous to the above, gives a unit vector from Q to E of $\begin{bmatrix} 1 - l/3 \\ -2l/3 \\ -10 - 2l/3 \end{bmatrix}$ (we leave this unnormalized for now, which is in this case allowed because we do not care about the factor k that will be in front of this vector as we see in the sequel), so we need to solve:

$$\begin{cases} 3 + l/3 + k(1 - l/3) & = x_{q1} \\ 4 + 2l/3 + k(-2l/3) & = y_{q1} \\ 5 + 2l/3 + k(-10 - 2l/3) & = 0 \end{cases} \tag{3}$$

which results in $Q_1 = (\frac{13}{2} - \frac{45}{l+15}, 9 - \frac{75}{l+15}, 0)$.

PART 3: SPHERES

EXERCISE 6

Given a sphere in $\mathbb{R}^3$ with radius $r = 3.14$ and center $C = (3, 5, 1)$

(a) What is the parametric representation of this sphere?
(b) Using the parametric representation, calculate the coordinates two opposing points on the sphere.
(c) Calculate the distance between these opposing points, and verify this using the diameter of the sphere.

Solutions:

(a) Implicit: $(x-3)^2 + (y-5)^2 + (z-1)^2 = 3.14^2$. Parametric: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + 3.14 \cdot \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$

(b) Full circle: 2π radians, so half a circle is π radians. Keep $\theta = \frac{\pi}{2}$, and take $\phi = 2\pi$ and $\phi_2 = \pi$. Filling in gives $P_1 = (6.14, 5, 1)$ and $P_2 = (-0.14, 5, 1)$.

(c) The distance between these points is $|P_2 - P_1| = \sqrt(6.28^2 + 0^2 + 0^2) = 6.28$. The diameter is $2r = 6.28$. They match, so it is correct.

EXERCISE 7

Given a sphere in $\mathbb{R}^3$ with center $C = (3, 3, 3)$ and radius 3.

(a) Find the range of parametric angles (θ, ϕ) that represents the part of the sphere viewed from $(3, 3, 3 - 3\sqrt{2})$
(b) Choose a point P on the sphere with $3 < z < 6$ and determine the the parametric form of the line that passes through this point and the origin of the sphere. Call this line l.
(c) Calculate the parametric angle (θ, ϕ) of the chosen point P with respect to the sphere.
(d) Calculate the intersection of l with the xy-plane.

Solutions:

(a) The parametric representation of the sphere is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 3 \cdot \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$. We are looking at it from the bottom (draw it for yourself). The first step is to calculate the lines tangent to this sphere, coming from the viewers point E. Calculation of tangent lines to a sphere means that the vector from the viewer to the sphere's surface is perpendicular to the vector from the sphere's center to the sphere's surface, that is:

$$\left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 3\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 - 3\sqrt{2} \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 3\begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}\right) = 0 \tag{4}$$

Using $\cos^2\alpha + \sin^2\alpha = 1$ gives $\cos\theta = -\frac{1}{\sqrt{2}}$, and all ϕ are possible. Using that ϕ is within the range $[0, 2\pi]$, and $\theta \in [0, \pi]$, we get $\theta = [\frac{3\pi}{4}, \pi]$ and $\phi \in [0, 2\pi]$.

(b) The implicit equation of the sphere is $(x-3)^2 + (y-3)^2 + (z-3)^2 = 3^2$. Take $P = (3, 3 + \sqrt{5}, 5)$ and check whether the implicit equation indeed holds. The unit vector from C to P is $(P - C)/|P - C| = \frac{1}{3}\begin{bmatrix} 0 \\ \sqrt{5} \\ 2 \end{bmatrix}$, so the line's equation is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 + \sqrt{5} \\ 5 \end{bmatrix} + \frac{t}{3} \cdot \begin{bmatrix} 0 \\ \sqrt{5} \\ 2 \end{bmatrix} \tag{5}$$

(c) This means we need to solve:

$$\begin{bmatrix}3\\3\\3\end{bmatrix} + 3\begin{bmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{bmatrix} = \begin{bmatrix}3\\3+\sqrt{5}\\5\end{bmatrix} \tag{6}$$

$$3\begin{bmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{bmatrix} = \begin{bmatrix}0\\\sqrt{5}\\2\end{bmatrix} \tag{7}$$

$$\begin{bmatrix}\sin\theta\cos\phi\\\sin\theta\sin\phi\\\cos\theta\end{bmatrix} = \begin{bmatrix}0\\\sqrt{5}/3\\2/3\end{bmatrix} \tag{8}$$

The third equation gives $\cos\theta = 2/3$, so $\theta \approx 48.2°$. The first equation gives $\sin\theta\cos\phi = 0$, which means that $\cos\phi = 0$ (because $\sin\theta \neq 0$). This leads to $\phi = 90°$.

(d) This means solving:

$$\begin{bmatrix}x\\y\\0\end{bmatrix} = \begin{bmatrix}3\\3+\sqrt{5}\\5\end{bmatrix} + \frac{t}{3}\cdot\begin{bmatrix}0\\\sqrt{5}\\2\end{bmatrix} \tag{9}$$

It follows that $t = -\frac{15}{2}$, $x = 3$ and $y = 3 - \frac{3}{2}\sqrt{5}$. So the point of intersection is $(3, 3 - \frac{3}{2}\sqrt{5}, 0)$.