# TUTORIAL 5 - SOLUTIONS

### MATRICES AND INTRODUCTIONTO TRANSFORMATIONS

## PART 1: THEORY

### EXERCISE 1

We start with some theoretical questions.

(a) What is a square matrix?
*A matrix with the same number of columns and rows.*

(b) What is the difference between a diagonal and a identity matrix?
*The identity matrix is a diagonal matrix where all the non-zero elements have value 1.*

(c) When can you sum two matrices? And when can you multiply them?
*Two matrices can summed (or subtract) only if they have the same dimensions (same numbers of rows and columns). They can be multiplied if the number of columns of the first one is the same as the number of rows of the second one.*

(d) Which conditions does a matrix need to satisfy in order to be able to be inverted?
*It needs to be a square matrix with a non zero determinant.*

(e) In class you have been shown that $A^{-1}A = AA^{-1} = I$, the identity matrix. Prove it for a generic 2×2 matrix $A$.
*Let us take a generic 2×2 matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*In order to write its inverse we need first to write the cofactor matrix as*

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

*then transpose it to get the adjoint matrix:*

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*finally we have to divide it by the determinant of A:*

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

*Now we just have to multiply it with A:*

$$A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

*The result is identical for $AA^{-1}$.*

(f) Does any matrix have a determinant?
*The determinant is defined for every square matrix.*

(g) What is a singular matrix? Which operation is not possible on singular matrices?
*A square matrix is singular if and only if its determinant is zero. It is not possible to invert a singular matrix, this is why non-singular matrices are also called invertible.*

(h) What is the cofactor of an element of a matrix?
*The cofactor of an element $a_{ij}$ is defined as:*

$$\text{cof}(a_{ij}) = (-1)^{i+j}\det(\text{minor}(a_{ij}))$$

*in other words, the cofactor is the determinant of the matrix without the $i-th$ row and the $j-th$ column, multiplied by $(-1)^{i+j}$*

# Part 2: Operations with matrices

## Exercise 2

Given the following matrices:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 5 & 2 \\ 3 & 2 & 1 \\ 5 & 3 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 4 & 4 & 4 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(a) Which matrices can be summed together?
(b) Compute those sums.
(c) Which matrices can be multiplied together? Does the order matter?
(d) Compute those multiplications.
(e) Compute the determinant for all matrices.
(f) Is it possible to compute the inverse of every matrix? If not, why?
(g) Compute the inverse matrices $A^{-1}$ and $B^{-1}$.

## Solutions

(a) $A$ and $D$, $E$ and $F$.

(b)

$$A + D = \begin{bmatrix} 3 & 8 & 3 \\ 6 & 5 & 3 \\ 6 & 4 & 2 \end{bmatrix} \qquad E + F = \begin{bmatrix} 2 & 4 & 4 & 4 \\ 3 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

(c) $CA$, $CD$, $BC$, $AD$, $DA$, $EF$, $FE$. The order does matter: $CA$ and $BC$ are allowed but $AC$ and $CB$ are not, since the dimensions do not match (see answer to question 1c)

(d)

$$CA = \begin{bmatrix} 11 & 12 & 11 \\ 25 & 29 & 18 \end{bmatrix} \quad CD = \begin{bmatrix} 22 & 18 & 4 \\ 29 & 36 & 13 \end{bmatrix} \quad BC = \begin{bmatrix} 17 & 22 & 11 \\ 11 & 16 & 13 \end{bmatrix} \quad AD = \begin{bmatrix} 16 & 19 & 7 \\ 22 & 27 & 9 \\ 14 & 13 & 3 \end{bmatrix} \quad DA = \begin{bmatrix} 19 & 20 & 15 \\ 13 & 16 & 9 \\ 19 & 24 & 11 \end{bmatrix}$$

$$EF = \begin{bmatrix} 1 & 4 & 4 & 4 \\ 3 & 0 & 2 & 1 \\ 1 & 4 & 6 & 5 \\ 4 & 4 & 8 & 6 \end{bmatrix} \qquad FE = \begin{bmatrix} 9 & 8 & 0 & 16 \\ 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 3 & 1 & 0 & 4 \end{bmatrix}$$

(e) $\det(A)$ = -4 $\quad\det(B)$ = -10 $\quad\det(D)$ = 20 $\quad\det(E)$ = 0 $\quad\det(F)$ = -12

It is not possible to define a determinant for matrix $C$ since it is not a square matrix.

(f) We cannot compute the inverse of matrix $C$ since it is not a square matrix and also of matrix $E$ since, as we saw before, its determinant is zero and therefore it is a singular matrix.

(g)

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -4 & 5 & -3 \\ 4 & -3 & 1 \\ 0 & -1 & 3 \end{bmatrix} \qquad B^{-1} = \frac{1}{10} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$$

## Exercise 3

Given the following matrices:

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 5 & 0 & 3 \\ 1 & 8 & 2 & 3 \end{bmatrix}$$

For each of them compute:

(a) The determinant
(b) The transpose matrix.
(c) The cofactor matrix.
(d) The adjoint matrix.
(e) The inverse matrix.
(f) For matrix $G$, you could have answered all the previous question without doing any calculations. Why?

**SOLUTIONS**

(a) $\det(G) = 1$, $\det(H) = -11$, $\det(J) = 12$, $\det(L) = 28$.

(b)

$$G^{\mathrm{T}} = G \qquad H^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} \qquad J^{\mathrm{T}} = \begin{bmatrix} 5 & 1 & 3 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 \end{bmatrix} \qquad L^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 0 & 2 & 5 & 8 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 3 & 3 \end{bmatrix}$$

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(c)

$$\operatorname{cof}(G) = G \qquad \operatorname{cof}(H) = \begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix} \qquad \operatorname{cof}(J) = \begin{bmatrix} 0 & 6 & -6 & 0 \\ 0 & -3 & 9 & 0 \\ 4 & -9 & 7 & 0 \\ -4 & 9 & -19 & 12 \end{bmatrix} \qquad \operatorname{cof}(L) = \begin{bmatrix} 8 & -4 & 6 & 4 \\ -10 & -16 & 24 & 30 \\ 4 & 12 & -32 & -12 \\ 0 & 0 & 14 & 0 \end{bmatrix}$$

(d)

$$\operatorname{adj}(G) = G \qquad \operatorname{adj}(H) = \begin{bmatrix} 1 & -3 \\ -4 & 1 \end{bmatrix} \qquad \operatorname{adj}(J) = \begin{bmatrix} 0 & 0 & 4 & -4 \\ 6 & -3 & -9 & 9 \\ -6 & 9 & 7 & -19 \\ 0 & 0 & 0 & 12 \end{bmatrix} \qquad \operatorname{adj}(L) = \begin{bmatrix} 8 & -10 & 4 & 0 \\ -4 & -16 & 12 & 0 \\ 6 & 24 & -32 & 14 \\ 4 & 30 & -12 & 0 \end{bmatrix}$$

(e)

$$G^{-1} = G \qquad H^{-1} = \begin{bmatrix} -\frac{1}{11} & \frac{3}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{1}{2} & \frac{3}{4} & \frac{7}{12} & -\frac{19}{12} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad L^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{5}{14} & \frac{1}{7} & 0 \\ -\frac{1}{7} & -\frac{4}{7} & \frac{3}{7} & 0 \\ \frac{3}{14} & \frac{6}{7} & -\frac{8}{7} & \frac{1}{2} \\ \frac{1}{7} & \frac{15}{14} & -\frac{3}{7} & 0 \end{bmatrix}$$

(f) Matrix G is the identity matrix of size 4. Its determinant is clearly 1 and all the other operations keep it identical.

# PART 3: GEOMETRICAL INTERPRETATION

### EXERCISE 4

Given the lists of vertices, compute the area of the following parallelograms:

$$A_1 = \{(3,1),(2,-3),(5,-2),(0,0)\} \qquad A_2 = \{(1,3),(3,1),(5,-1),(7,-3)\}$$

$$A_3 = \{(-1,-3),(4,-2),(0,-6),(5,-5)\} \qquad A_4 = \{(1,5),(-1,9),(2,9),(4,5)\}$$

**SOLUTIONS**

Let us choose the first point $(3,1)$ as our starting point (note: any starting point is fine). Therefore the two vectors will be:

$$\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

and the matrix will be:

$$A_1 = \begin{bmatrix} -3 & 2 \\ -1 & -3 \end{bmatrix}$$

the absolute value of determinant of which is $|\det(A_1)| = 11$, which means that the area is 11.

Similarly for the other parallelograms:

$$A_2 \;:\; \vec{u} = \begin{bmatrix} 5-1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 3-1 \\ 1-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}\right)\right| = 0$$

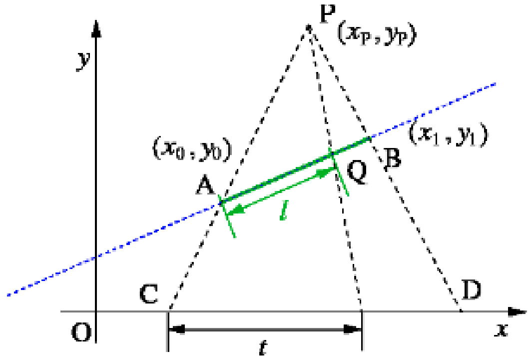
$$A_3 \;:\; \vec{u} = \begin{bmatrix} 5-0 \\ -5+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1-0 \\ -3+6 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\right)\right| = 16$$

$$A_4 \;:\; \vec{u} = \begin{bmatrix} 4-1 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1-1 \\ 9-5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix}\right)\right| = 12.$$

**Exercise 5**




Given the list of vertices, compute the volume of the following parallelepipeds.

$$\pi_1 = \{(3,1,0),(2,-3,0),(5,-2,0),(0,0,-4)\} \qquad \pi_2 = \{(1,7,2),(3,7,1),(5,7,0),(-1,-1,-1)\}$$
$$\pi_3 = \{(2,1,5),(2,-1,9),(2,2,9),(-3,4,5)\} \qquad \pi_4 = \{(-1,2,-3),(4,2,-2),(0,2,-6),(5,-1,-5)\}$$

*The points are ordered $\pi = \{a,b,c,d\}$ and connected as in figure.*

**Solutions**

Similarly as before, we take a point and compute the distance vector to the three points connected to it.

$$\vec{u}_1 = \begin{bmatrix} 2-3 \\ -3-1 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 5-3 \\ -2-1 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0-3 \\ 0-1 \\ -4-0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -4 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} -1 & 2 & -3 \\ -4 & -3 & -1 \\ 0 & 0 & -4 \end{bmatrix}\right)\right| = 44$$

Similarly for the other parallelopipeds:

$\pi_2$ :

$$\vec{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} -2 \\ -8 \\ -3 \end{bmatrix}$$

$$\det(\pi_2) = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 0 & -8 \\ -1 & -2 & -3 \end{bmatrix} = 0$$

$\pi_3$ :

$$\vec{u}_1 = \begin{bmatrix} 0 \\ -2 \\ 4 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} 0 & 0 & -5 \\ -2 & 1 & 3 \\ 4 & 4 & 0 \end{bmatrix}\right)\right| = 60$$

$\pi_4$ :

$$\vec{u}_1 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$$

$$\left|\det\left(\begin{bmatrix} 5 & 1 & 6 \\ 0 & 0 & -3 \\ 1 & -3 & -2 \end{bmatrix}\right)\right| = 48$$

**Exercise 6**

Two of the figures in the previous exercises have a zero oriented area or volume.

(a) What is the geometrical meaning of having a zero oriented area or volume?
(b) What restriction can you put on the vertices in order to avoid this case?

**Solutions**

(a) For a parallelogram the area is zero when two "edges" are parallel to each other and therefore no paralleogram can be formed to start with.

For a parallelopiped the volume is zero when any one of the edges, say $\vec{u}_3$ can be expressed as $\lambda\vec{u}_1 + \mu\vec{u}_2$, with $\lambda$ and $\mu$ two scalar multipliers. In that case too, no parallelepiped can be formed.

(b) As long as the conditions in (a) are not satisfied.