TUTORIAL 6 - EXERCISES

MATRIX RELOADED 1 - TRANSFORMATIONS

PART 1: THEORY

EXERCISE 1

We start with some theoretical questions.

(a) What is the difference between active and passive transformation? Sketch examples.
(b) Name 5 different kinds of matrix transformations that have taught in the lecture. Indicate whether these transformations are active or passive (give at least one of each).
(c) Given: matrix M that transforms $\vec{a}$ into $\vec{b}$. Use the convention of the lecture; how do $M, \vec{a}$ and $\vec{b}$ relate to each other?
(d) When calculating matrix transformations, people often add a fictitious dimension $\hat{f}$, so that 2D becomes (2+1)D, for example. Explain in your own words why this is necessary. If you find this difficult, try to start with an example of point translation.
(e) Say we have a real vector in 2D: $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and we add a third dimension $\hat{f} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ that we call the fictitious dimension. What do we know about the dot product of these two? Explain in your own words why this is the case.
(f) How do you call a square matrix that does not have an inverse?

PART 2: NOTATIONS AND GENERAL FORMULAE

In general, when one would like to apply a particular transformation on a point P, one could draw a vector $\vec{v}$ from the origin to P, and apply the specified transformation matrix M on it. This results in a following formula:

$$\vec{w}' = M\vec{v}' \tag{1}$$

with $\vec{w}$ the resulting transformed vector, and $\vec{w}'$ and $\vec{v}'$ denote the vectors $\vec{w}$ and $\vec{v}$ with an added fictitious dimension (see 1d). If the transformation considers a translation, M and $\vec{w}$ get a subscript t (i.e., $M_t, \vec{w}_t$). The same holds for projections (p), reflections (r), etcetera. We will use this notation convention in the tutorial.

Often, $\vec{w}$ can also be calculated explicitly using knowledge from earlier lectures. These transformation matrices are just often an extra tool to consider when doing linear operations. Sometimes the answer is much easier to write down using your intuition than using transformation matrices (for example when considering translation or 2D projections on the x-axis).

EXERCISE 2 - GENERAL FORMULAE

Consider linear operations in 3D (+1D fictitious). Do the following exercises for translation, projection, reflection, scaling, shearing and rotation of a point P. Assume for projection and reflection that you are doing this over a line/plane through the origin.

(a) Sketch an example. Include $\vec{w}$ and $\vec{v}$ in your drawing.
(b) What is the general formula for the transformation matrix M? For shear and rotation, choose the form of M that is coherent with your drawing.
(c) Give the general formula for $\vec{w}$; do this only for translation, project and reflection. (In the case of projection and reflection, use the normal $\hat{n}$ going through the line that is projected/reflected on and points towards P.)

PART 3: EXAMPLES

EXERCISE 3 - TRANSLATION

Given point P (1,2,3).

(a) Say we would like to translate point P by 3 in the x-direction, 1 in the y-direction and 6 in the z-direction. Call this direction vector $\vec{a}$. Give $M_t(\vec{a})$ and calculate $\vec{w}_t$ using the matrix.
(b) How can you check your answer in (a) without using a transformation matrix?
(c) Draw the translation calculated in (b).

EXERCISE 4 - PROJECTION OF A POINT

Given a point A (1,2).

(a) Calculate and draw the projection of point A on the x-axis, without doing any calculations.
(b) Now write down the projection matrix M_p (use your answer in (c)). Calculate the projection of point A from this (this follows from $\vec{w}_p$). Does this indeed coincide with your answer in (e)?

Now consider B (4,1), C (1,1) and D (5,5)

(c) Write down the parametric equation of the line L through C and D.

(d) Determine the normal vector that is perpendicular to line L, and points towards point B.

(e) Calculate the projection of B on line L without using a transformation matrix. Make a drawing.

(f) Calculate the projection of B on line L while using a transformation matrix.

EXERCISE 5- REFLECTION OF A POINT

Given a point A (2,3), B (-1,2) and C (-2,4))

(a) Write down the parametric equation of the line L through B and C.
(b) Determine the normal vector that is perpendicular to line L, and points towards point A.
(c) Calculate the reflection of A over line L without using a transformation matrix. Make a drawing.
(d) Calculate the reflection of A over line L while using a transformation matrix.

EXERCISE 6- REFLECTION OF A VECTOR

Given a vector in 2D $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ starting at point $A = (3,4)$. Furthermore, given are $B = (0,1)$ and $C = (3,2)$.

(a) Write down the slope-intersect form of the line L through B and C.
(b) Calculate the reflection of $\vec{v}$ over line L.

EXERCISE 7 - SCALING

Given point P (2,1,3) and point Q (1,2,3) in $\mathbb{R}^3$, O being the origin.

(a) Scale the vector $\overrightarrow{OP}$ with a uniform scaling of 3. Use the transformation matrix M_{sc}.
(b) Scale the vector $\overrightarrow{OQ}$ with a scaling of $(s_x, s_y, s_z) = (2,1,1)$.

EXERCISE 8 - SHEARING

Given point P (2,3) (in 2D). Consider the rectangle R that is made up by the x-axis, the y-axis and point P as right-upper corner.

(a) Apply the matrix $M_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to all corners of R. Draw the result for $b = 1,2,3$.
(b) Say our rectangle is fixed (attached) onto the x-axis, but can move through and away from the y-axis (like pudding). What does the matrix M_{sh} look like?
(c) Now say we would like to fix the corners of R to the y-axis, but move P towards (2,-2). Calculate M_{sh} for this scenario.

EXERCISE 9- ROTATION

Consider point P (1,2) in 2D. The rotations in this exercise are considered active rotations.

(a) Rotate P counterclockwise by 90°.
(b) Rotate P clockwise by 31°.

Now consider 3D rotation. Consider point Q (1,2,3)

(c) Rotate Q counterclockwise by 90°around the x axis.
(d) Rotate Q counterclockwise by 31°around the y axis.

PART 4: LINEAR TRANSFORMATIONS AND COMBINING THEM

EXERCISE 10 - LINEAR TRANSFORMATIONS

In this exercise, do all linear transformations using transformation matrices M. Given points P (1,2), Q (4,1) and R (2,2). Call the vectors from the origin to these points $\vec{p}, \vec{q}$ and $\vec{r}$, respectively. These are 2D vectors (without the +1D fictitious dimension; but probably you will need to add that in your calculations).

(a) Rotate P by 35°counterclockwise.
(b) Rotate Q by 35°counterclockwise.
(c) Rotate the vectors $\vec{p} + \vec{q}$ by 35°counterclockwise.
(d) Add your answers of (a) and (b) together and compare them to (c). What do you notice? Can you think of a general form of a criterium that linear point transformations hold, related to this?
(e) Scale P by 3 uniformly, and rotate by 35°counterclockwise.
(f) Rotate P by 35°counterclockwise and multiply the result by 3. What do you notice when comparing this to your answer in (e)? Can you think of a general form of a criterion that linear point transformations hold, related to this?

In scalar multiplication (or addition), we have the so-called *commutative* property, which means that we can swap the order of the scalars within one operation: $a * b = b * a$. Let's check for some matrix transformation whether this holds there, too. This would mean that for two transformations T_a and T_b and a vector $\vec{v}$, the following equality holds: $T_a(T_b(\vec{v})) = T_b(T_a(\vec{v}))$.

(g) Rotate P by $35°$counterclockwise and translate it by 6 in the x-direction. Now do it the other way around. Do these operations commute?

(h) Reflect Q over the line with slope-intersect equation $y = x$, and then translate it by 3 in the y-direction. Now do it the other way around. Do these operations commute?

(i) Scale R by 2 (uniformly), and rotate by $5°$counterclockwise. Now do it the other way around. Do these operations commute?

EXERCISE 11- COMBINING TRANSFORMATIONS

Given point P (1,2,1), R (0,1,3) and Q (-1,3,2). Furthermore, given vector $\vec{a} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Rotations are counterclockwise and around the z-axis. Call the vectors from the origin towards the points P, Q and R: $\vec{p}, \vec{q}$ and $\vec{r}$, respectively. Furthermore, given are the planes:

$$\begin{aligned} A_1 &: \quad 2x + y + z = 0 \\ A_2 &: \quad -x - 5y + z = 0 \\ A_3 &: \quad x + 2y - z = 0 \end{aligned}$$

(a) Translate Q by $\vec{a}$ and then reflect over A_1.
(b) Project P on A_2, and then rotate by $10°$.
(c) Reflect R over A_3 and rotate the result by $45°$.
(d) Scale R with 3 uniformly, and reflect over A_3, and rotate the result by $45°$(Hint: use ideas from exercise 10)
(e) Reflect Q over A_3, and rotate the result by $45°$.
(f) Reflect $\vec{r} + \vec{q}$ over A_3, and rotate the transformed endpoint (not the vector) by $45°$(Hint: use ideas from exercise 10)
(g) Rotate your answer in question (e) by -45°.

EXERCISE 12 - COORDINATE TRANSFORMATIONS

Given point P (3,1) in the regular Cartesian coordinate system in 2D. Rotations are counterclockwise in this exercise.

(a) Rotate P by $35°$. Call this point Q. Draw this. Are P and Q the same point on your paper?
(b) Rotate the x and y axes by $35°$. Call P in this coordinate system R. Give the coordinates of R and draw it. Are P and R the same point on your paper?
(c) Which of the two transformations applied in (a) and (b) do we call 'active' and which do we call 'passive' transformations?