TUTORIAL 6 - SOLUTIONS

MATRIX RELOADED 1 - TRANSFORMATIONS

PART 1: THEORY

EXERCISE 1

We start with some theoretical questions.

(a) What is the difference between active and passive transformation? Sketch examples.
(b) Name 5 different kinds of matrix transformations that have taught in the lecture. Indicate whether these transformations are active or passive (give at least one of each).
(c) Given: matrix M that transforms $\vec{a}$ into $\vec{b}$. Use the convention of the lecture; how do $M, \vec{a}$ and $\vec{b}$ relate to each other?
(d) When calculating matrix transformations, people often add a fictitious dimension $\hat{f}$, so that 2D becomes (2+1)D, for example. Explain in your own words why this is necessary. If you find this difficult, try to start with an example of point translation.
(e) Say we have a real vector in 2D: $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and we add a third dimension $\hat{f} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ that we call the fictitious dimension. What do we know about the dot product of these two? Explain in your own words why this is the case.
(f) How do you call a square matrix that does not have an inverse?

Solutions:

(a) In active transformation, the transformation is done on an object (e.g., a point). In passive transformation, the transformation is done on the axes (or in graphical terms: the camera), leaving the objects fixed in their physical position.
(b) Transformations like translation, projection, reflection, scaling, shearing, rotation all can be active or passive. Active transformation means that it is applied on an object (like a point). Passive transformation means that it is applied on a coordinate system.
(c) $\vec{b} = M \cdot \vec{a}$
(d) The fictitious dimension is needed to include constant (scalar) terms in the transformation. For example, if you would like a translation of a point by (a_x, a_y, a_z), you need a transformation matrix of $\begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$
(e) Then $\vec{v} \cdot \hat{f} = 0$, always, because $\hat{f}$ only has nonzero elements in its own (fictitious) dimension, while $\vec{v}$ only has nonzero elements in real dimensions. This means they are effectively orthogonal to each other, hence inner product 0.
(f) A singular matrix

PART 2: NOTATIONS AND GENERAL FORMULAE

In general, when one would like to apply a particular transformation on a point P, one could draw a vector $\vec{v}$ from the origin to P, and apply the specified transformation matrix M on it. This results in a following formula:

$$\vec{w'} = M\vec{v'} \tag{1}$$

with $\vec{w}$ the resulting transformed vector, and $\vec{w'}$ and $\vec{v'}$ denote the vectors $\vec{w}$ and $\vec{v}$ with an added fictitious dimension (see 1d). If the transformation considers a translation, M and $\vec{w}$ get a subscript t (i.e., $M_t, \vec{w}_t$). The same holds for projections (p), reflections (r), etcetera. We will use this notation convention in the tutorial.

Often, $\vec{w}$ can also be calculated explicitly using knowledge from earlier lectures. These transformation matrices are just often an extra tool to consider when doing linear operations. Sometimes the answer is much easier to write down using your intuition than using transformation matrices (for example when considering translation or 2D projections on the x-axis).

EXERCISE 2 - GENERAL FORMULAE

Consider linear operations in 3D (+1D fictitious). Do the following exercises for translation, projection, reflection, scaling, shearing and rotation of a point P. Assume for projection and reflection that you are doing this over a line/plane through the origin.

(a) Sketch an example. Include $\vec{w}$ and $\vec{v}$ in your drawing.
(b) What is the general formula for the transformation matrix M? For shear and rotation, choose the form of M that is coherent with your drawing.
(c) Give the general formula for $\vec{w}$; do this only for translation, project and reflection. (In the case of projection and reflection, use the normal $\hat{n}$ going through the line that is projected/reflected on and points towards P.)

Solutions for translation:

(a) Sketch not included here. $\vec{v}$ is the vector from the origin to point P. $\vec{w}$ is the vector from the origin to the transformed point, i.e., the transformed vector. (This answer does not change for other transformations below).

(b) $M_t = \begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\vec{w}_t = \vec{v} + \vec{a}$

Solutions for projection:

(b) Given a normal $\hat{n}$ to the line you are projecting on. $M_p = \begin{bmatrix} 1-n_x^2 & -n_x n_y & -n_x n_z & 0 \\ -n_x n_y & 1-n_y^2 & -n_y n_z & 0 \\ -n_x n_z & -n_y n_z & 1-n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\vec{w} = \vec{v} - (\vec{v} \cdot \hat{n})\hat{n}$

Solutions for reflection:

(b) $M_r = \begin{bmatrix} 1-2n_x^2 & -2n_x n_y & -2n_x n_z & 0 \\ -2n_x n_y & 1-2n_y^2 & -2n_y n_z & 0 \\ -2n_x n_z & -2n_y n_z & 1-2n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\vec{w} = \vec{v} - 2(\vec{v} \cdot \hat{n})\hat{n}$

Solutions for scaling:

(b) $M_{sc} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) Situationally dependent.

Solutions for shearing:

(b) For a shear in the x direction that scales with z: $M_{sh} = \begin{bmatrix} 1 & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) Situationally dependent.

Solutions for rotation:

(b) For anti-clockwise rotation around the x axis by an angle of θ: $M_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) Situationally dependent.

PART 3: EXAMPLES

For the record: in the sequel, $\vec{w}$ is now including the fictitious dimension. In the above we have made a distinction between $\vec{w}'$ (3+1D) and $\vec{w}$ (3D), which we will drop now for convenience, because it always has value 1 in the fictitious dimension.

EXERCISE 3 - TRANSLATION

Given point P (1,2,3).

(a) Say we would like to translate point P by 3 in the x-direction, 1 in the y-direction and 6 in the z-direction. Call this direction vector $\vec{a}$. Give $M_t(\vec{a})$ and calculate $\vec{w}_t$ using the matrix.
(b) How can you check your answer in (a) without using a transformation matrix?
(c) Draw the translation calculated in (b).

Solutions:

(a) $M_t(\vec{a}) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Call $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ the 3+1D vector from the origin to P. Then $\vec{w}_t = M_t \cdot \vec{v} = \begin{bmatrix} 4 \\ 3 \\ 9 \\ 1 \end{bmatrix}$. So the transformed point would be $(4,3,9)$.

(b) Yes, use $\vec{w}_t = \vec{v} + \vec{a} = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$, resulting in the transformed point $(4,3,9)$.

(c) *

Exercise 4 - Projection of a point

Given a point A (1,2).

(a) Calculate and draw the projection of point A on the x-axis, without doing any calculations.
(b) Now write down the projection matrix M_p. Calculate the projection of point A from this (this follows from $\vec{w}_p$). Does this indeed coincide with your answer in (a)?

Solutions:

(a) The coordinates of A's projection on the x-axis is simply putting its other coordinates to zero: $(1,0)$.

(b) $M_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then calculate $\vec{w}_p = M_p \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So the projected coordinate is $(1,0)$. It indeed coincides!

Now consider B (4,1), C (1,1) and D (5,5)

(c) Write down the parametric equation of the line L through C and D.

(d) Determine the normal vector that is perpendicular to line L, and points towards point B.

(e) Calculate the projection of B on line L without using a transformation matrix. Make a drawing.

(f) Calculate the projection of B on line L while using a transformation matrix.

Solutions:

(c) $L: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{t}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) Note that L goes through the origin, so we can use the formulae from the lecture. Perpendicular to L means that $\hat{n} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$. So $n_x + n_y = 0$, or $n_x = -n_y$. There are two normalized vectors that hold this; we need the one that points right down (draw and you'll see that indeed B is right-down of its projection on L), so that $n_x > 0$ and $n_y < 0$. This means $\hat{n} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(e) Call $\vec{v}$ the vector from the origin to B. We use $\vec{w} = \vec{v} - (\vec{v} \cdot \hat{n})\hat{n} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}$, giving the projected point $(\frac{5}{2}, \frac{5}{2})$. Check that this point lies indeed on line L as you defined it in (c).

(f) The transformation matrix M_p equals: $M_p = \begin{bmatrix} 1-n_x^2 & -n_x n_y & 0 \\ -n_x n_y & 1-n_y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1-\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Now, calculate $\vec{w}_p = M_p \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \\ 1 \end{bmatrix}$. So the projected point is $(\frac{5}{2}, \frac{5}{2})$. Indeed, this is the same as in (e).

Exercise 5- Reflection of a point

Given a point A (2,3), B (-1,2) and C (-2,4))

(a) Write down the parametric equation of the line L through B and C.
(b) Determine the normal vector that is perpendicular to line L, and points towards point A.
(c) Calculate the reflection of A over line L without using a transformation matrix. Make a drawing.
(d) Calculate the reflection of A over line L while using a transformation matrix.

Solutions:

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \frac{t}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(b) Note that L goes through the origin, so we can use the formulae from the lecture. A normal would be such that $\hat{n} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$, so $n_x = 2n_y$. A normal on L towards A means $n_x, n_y > 0$ (draw and you will see). So there is one possibility of such a normalized normal vector: $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(c) We use $\vec{w} = \vec{v} - 2 \cdot (\vec{v} \cdot \hat{n})\hat{n} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2 \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 28/5 \\ 14/5 \end{bmatrix} = \begin{bmatrix} -18/5 \\ 1/5 \end{bmatrix}$. So the reflected point is $(\frac{-18}{5}, \frac{1}{5})$. Note that this sounds reasonable if you sketch it.

(d) The transformation matrix is $M_r = \begin{bmatrix} 1-2n_x^2 & -2n_x n_y & 0 \\ -2n_x n_y & 1-2n_y^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & -4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\vec{w}_r = M_r \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18/5 \\ 1/5 \\ 1 \end{bmatrix}$. So the reflected point is again $(\frac{-18}{5}, \frac{1}{5})$.

Exercise 6- Reflection of a vector

Given a vector in 2D $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ starting at point $A = (3,4)$. Furthermore, given are $B = (0,1)$ and $C = (3,2)$.

(a) Write down the slope-intersect form of the line L through B and C.
(b) Calculate the reflection of $\vec{v}$ over line L.

Solutions:

(a) Start with $y = ax + b$. The unit vector from B to C is $\vec{d} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so the slope a is $\frac{1}{3}$, so we have $y = \frac{1}{3}x + b$. Then we find b by filling in the point B that was given, which results in $b = 1$, so $y = \frac{1}{3}x + 1$. Note that this line does not pass through the origin.

(b) The normalized line normal to L in the direction of A is $\hat{n} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Reflecting a vector only needs the normal the the line to calculate the perpendicular component of the vector (which we then substract twice to get the transformed vector): $\vec{w} = \vec{v} - 2(\vec{v} \cdot \hat{n})\hat{n} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{2}{10}(3) \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 36/10 \\ 2/10 \end{bmatrix}$. You can check whether the magnitude of the transformed vector $\vec{w}$ is the same as that of $\vec{v} = \sqrt{13}$. Note: it would be instructive to work out this exercise for a different starting point A.

Exercise 7 - Scaling

Given point P (2,1,3) and point Q (1,2,3) in $\mathbb{R}^3$, O being the origin.

(a) Scale the vector $\overrightarrow{OP}$ with a uniform scaling of 3. Use the transformation matrix M_{sc}.
(b) Scale the vector $\overrightarrow{OQ}$ with a scaling of $(s_x, s_y, s_z) = (2,1,1)$.

Solutions:

(a) The transformation matrix is $M_{sc} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then the scaled point P is calculated by $w_{sc} = M_{sc} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 9 \\ 1 \end{bmatrix}$. So the corresponding point is $(6,3,9)$.

(b) The transformation matrix is $M_{sc} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then the scaled point Q is calculated by $w_{sc} = M_{sc} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$. So the corresponding point is $(2,2,3)$.

Exercise 8 - Shearing

Given point P (2,3) (in 2D). Consider the rectangle R that is made up by the x-axis, the y-axis and point P as right-upper corner.

(a) Apply the matrix $M_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to all corners of R. Draw the result for $b = 1,2,3$.
(b) Say our rectangle is fixed (attached) onto the x-axis, but can move through and away from the y-axis (like pudding). What does the matrix M_{sh} look like?
(c) Now say we would like to fix the corners of R to the y-axis, but move P towards (2,-2). Calculate M_{sh} for this scenario.

Solutions:

(a) The corners are (2,3), (2,0), (0,3) and (0,0). Applying M_{sh} implies the multiplication of this matrix to $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ (and so on). Applying to all these corners gives (respectively): $(2, 2b+3)$, $(2, 2b)$, $(0,3)$ and $(0,0)$. Filling in $b = 1,2,3$ gives the desired points.

(b) This means that we shear in the y direction, which implies the following form of M_{sh}: $\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) This exercise performs a y shear, which means that all points $(0, a)$ remain the same. This implies that the second column of the M_{sh} should be $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. So the matrix is of the form: $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Searching for the right value of b gives that we state $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = M_{sh} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. This results in $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2b+3 \\ 1 \end{bmatrix}$. This results in $b = \frac{-5}{2}$. So $M_{sh} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

EXERCISE 9- ROTATION

Consider point P (1,2) in 2D. The rotations in this exercise are considered active rotations.

(a) Rotate P counterclockwise by 90°.
(b) Rotate P clockwise by 31°.

Solutions:
Note that the angle θ in general formulae for the transformation matrices is taken counterclockwise. So if a *clockwise* rotation with angle a is asked, just use $\theta = -a$.

(a) The transformation matrix is $M_{ro} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The result is then $\vec{w}_{ro} = M_{ro}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. So rotated coordinate is $(-2, 1)$

(b) The transformation matrix is for **clockwise** rotation is $M_{ro} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.86 & -0.52 & 0 \\ 0.52 & 0.86 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The rotation is then $\vec{w}_{ro} = M_{ro}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.2 \\ 2.2 \\ 1 \end{bmatrix}$. So rotated coordinate is $(-0.2, 2.2)$.

Now consider 3D rotation. Consider point Q (1,2,3)

(c) Rotate Q counterclockwise by 90°around the x axis.
(d) Rotate Q counterclockwise by 31°around the y axis.

Solutions:

(c) The transformation matrix is $M_{ro} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The answer is $(1, -3, 2)$.

(d) The transformation matrix is $M_{ro} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The answer is $\approx (-2.4, 2, 2.1)$.

PART 4: LINEAR TRANSFORMATIONS AND COMBINING THEM

EXERCISE 10 - LINEAR TRANSFORMATIONS

In this exercise, do all linear transformations using transformation matrices M. Given points P (1,2), Q (4,1) and R (2,2). Call the vectors from the origin to these points $\vec{p}, \vec{q}$ and $\vec{r}$, respectively. These are 2D vectors (without the +1D fictitious dimension; but probably you will need to add that in your calculations).

(a) Rotate P by 35°counterclockwise.
(b) Rotate Q by 35°counterclockwise.
(c) Rotate the vector $\vec{p} + \vec{q}$ by 35°counterclockwise.
(d) Add your answers of (a) and (b) together and compare them to (c). What do you notice? Can you think of a general form of a criterium that linear point transformations hold, related to this?
(e) Scale P by 3 uniformly, and rotate by 35°counterclockwise.
(f) Rotate P by 35°counterclockwise and multiply the result by 3. What do you notice when comparing this to your answer in (e)? Can you think of a general form of a criterion that linear point transformations hold, related to this?

Solutions:
Note: for the general formulae of the transformation matrices used, please look at similar questions above.
Another note: in these exercises, where we do transformations on points, we actually ask for the resulting *point*. In these solutions, although for simplicity we may sometimes only state the resulting $\vec{w}$, but remember that a vector is not the same as a point. The resulting point can be easily retrieved by taking the endpoint of $\vec{w}$, having coordinates equal to its elements (without the fictitious dimension, if that is used). For example, if $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ (in 3+1D), the resulting point is $(1, 2, 3)$, and if $\vec{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ (in 3D, without fictitious dimension), the resulting point is $(4, 5, 6)$. Sometimes, it is explicitly asked to transform a *vector* (NOT a point), like in Exercise 6.

(a) $\begin{bmatrix} -0.3 \\ 2.2 \end{bmatrix}$
(b) $\begin{bmatrix} 2.7 \\ 3.1 \end{bmatrix}$
(c) $\vec{p} + \vec{q} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Rotating this gives the vector $\begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix}$.
(d) They are equal. This points towards the additive property of linear transformations: $T(a) + T(b) = T(a + b)$. This property applies on matrices!

(e) $3\vec{p} = \begin{bmatrix}3\\6\end{bmatrix}$. Rotating this gives the vector $\begin{bmatrix}-0.99\\6.64\end{bmatrix}$.

(f) They are equal. This points towards the multiplicative property of linear transformations: $T(\lambda a) = \lambda \cdot T(a)$, with λ being a scalar.This property applies on matrices!

In scalar multiplication (or addition), we have the so-called *commutative* property, which means that we can swap the order of the scalars within one operation: $a * b = b * a$. Let's check for some matrix transformation whether this holds there, too. This would mean that for two transformations T_a and T_b and a vector $\vec{v}$, the following equality holds: $T_a(T_b(\vec{v})) = T_b(T_a(\vec{v}))$.

(g) Rotate P by 35°counterclockwise and translate it by 6 in the x-direction. Now do it the other way around. Do these operations commute?

(h) Reflect Q over the line with slope-intersect equation $y = x$, and then translate it by 3 in the y-direction. Now do it the other way around. Do these operations commute?

(i) Scale R by 2 (uniformly), and rotate by 5°counterclockwise. Now do it the other way around. Do these operations commute?

Solutions:

(g) The rotation of P is already calculated in (a): $\begin{bmatrix}-0.3\\2.2\end{bmatrix}$. Translating this by 6 in the x-direction gives $\begin{bmatrix}5.7\\2.2\end{bmatrix}$. Now the other way around: first we translate $\vec{p}$ to be $\begin{bmatrix}7\\2\end{bmatrix}$. Now we rotate. This gives $\begin{bmatrix}4.6\\5.6\end{bmatrix}$. Unequal to $\begin{bmatrix}5.7\\2.2\end{bmatrix}$, so they do not commute.

(h) Reflection we do using $\vec{w}_r = \vec{q} - 2(\vec{q} \cdot \hat{n})\hat{n}$. The normalized normal vector to that line (in the direction of Q) is $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$. So reflection gives $\begin{bmatrix}1\\4\end{bmatrix}$ (indeed, the coordinates swap as you reflect across the line $y = x$). Now translate, this gives a final result of $\begin{bmatrix}1\\7\end{bmatrix}$. Now the other way around: translating Q gives $\begin{bmatrix}4\\4\end{bmatrix}$. Reflecting this gives $\begin{bmatrix}4\\4\end{bmatrix}$ again, because this lays along the $y = x$ line. Moreover, this is not equal to $\begin{bmatrix}1\\7\end{bmatrix}$, so they do not commute.

(i) Scaling gives $\begin{bmatrix}4\\4\end{bmatrix}$. Rotating gives $\begin{bmatrix}3.64\\4.33\end{bmatrix}$. Then, we rotate $\vec{r}$ first, which results in $\begin{bmatrix}1.818\\2.167\end{bmatrix}$. Scaling this also gives $\begin{bmatrix}3.64\\4.33\end{bmatrix}$, so they commute! This is a consequence of the scaling transformation being uniform.

EXERCISE 11 - COMBINING TRANSFORMATIONS

Given point P (1,2,1), R (0,1,3) and Q (-1,3,2). Furthermore, given vector $\vec{a} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$. Rotations are counterclockwise and around the z-axis. Call the vectors from the origin towards the points P, Q and R: $\vec{p}, \vec{q}$ and $\vec{r}$, respectively. Furthermore, given are the planes:

$$\begin{aligned} A_1: &\quad 2x + y + z = 0 \\ A_2: &\quad -x - 5y + z = 0 \\ A_3: &\quad x + 2y - z = 0 \end{aligned}$$

(a) Translate Q by $\vec{a}$ and then reflect over A_1.
(b) Project P on A_2, and then rotate by 10°.
(c) Reflect R over A_3 and rotate the result by 45°.
(d) Scale R with 3 uniformly, and reflect over A_3, and rotate the result by 45°(Hint: use ideas from exercise 10)
(e) Reflect Q over A_3, and rotate the result by 45°.
(f) Reflect $\vec{r} + \vec{q}$ over A_3, and rotate the transformed endpoint (not the vector) by 45°(Hint: use ideas from exercise 10)
(g) Rotate your answer in question (e) by -45°.

Solutions:

(a) The translation gives $\vec{v} = \begin{bmatrix}-2\\4\\4\end{bmatrix}$. Now, we find the normal vector to A_1: $\hat{n} = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}$. Then we use $\vec{w}_r = \vec{v} - 2(\vec{v} \cdot \hat{n})\hat{n} = \begin{bmatrix}-14/3\\8/3\\8/3\end{bmatrix}$.

(b) We find that $\hat{n} = \begin{bmatrix}-1\\-5\\1\end{bmatrix}\frac{1}{\sqrt{27}}$. Then we use $\vec{w}_r = \vec{p} - (\vec{p} \cdot \hat{n})\hat{n} = \begin{bmatrix}17/27\\4/27\\37/27\end{bmatrix}$. Now we rotate by 10°using the rotation matrix, to get $\begin{bmatrix}0.59\\0.25\\1.37\end{bmatrix}$.

(c) We find that $\hat{n} = \begin{bmatrix}1\\2\\-1\end{bmatrix}\frac{1}{\sqrt{6}}$. Then we use $\vec{w}_r = \vec{r} - 2(\vec{r} \cdot \hat{n})\hat{n} = \frac{1}{3}\begin{bmatrix}1\\5\\8\end{bmatrix}$. Rotation gives then a resulting vector of $\frac{1}{3\sqrt{2}}\begin{bmatrix}-4\\6\\8\sqrt{2}\end{bmatrix}$.

(d) Remember that $T(\lambda\vec{a}) = \lambda T(\vec{a})$ where T is a linear transformation and λ a scalar. This means that we can multiply the answer in (c) by 3: $\vec{w}_r = \frac{1}{\sqrt{2}}\begin{bmatrix}-4\\6\\8\sqrt{2}\end{bmatrix}$.

(e) We find that $\hat{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \frac{1}{\sqrt{6}}$. Then we use $\vec{w}_r = \vec{q} - 2(\vec{q} \cdot \hat{n})\hat{n} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Rotation gives then a resulting vector of $\begin{bmatrix} -\frac{1}{\sqrt{2}} - \sqrt{2} \\ \frac{1}{\sqrt{2}} - \sqrt{2} \\ 3 \end{bmatrix}$.

(f) Remember that $T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b})$, so we can simply add the answers of (c) and (e), which becomes $\frac{1}{3\sqrt{2}} \begin{bmatrix} -13 \\ 3 \\ 17\sqrt{2} \end{bmatrix}$.

(g) Rotating by a negative number in this exercise means that we rotate clockwise, so reverting the rotation we did in question (e) again gives $Q = (-2, 1, 3)$. More formally, you can use the fact that rotatting by 45°and then rotating by -45°are each other's inverse: $T_{-45}(T_{45}(\vec{q}) = T_{45}^{-1}(T_{45}(\vec{q})) = \vec{q}$ (with T_θ the rotation transformation by angle θ).

Exercise 12 - Coordinate transformations

Given point P (3,1) in the regular Cartesian coordinate system in 2D. Rotations are counterclockwise in this exercise.

(a) Rotate P by 35°. Call this point Q. Draw this. Are P and Q the same point on your paper?
(b) Rotate the x and y axes by 35°. Call P in this coordinate system R. Give the coordinates of R and draw it. Are P and R the same point on your paper?
(c) Which of the two transformations applied in (a) and (b) do we call 'active' and which do we call 'passive' transformations?

Solutions:

(a) Add a fictitious dimension, to get $\vec{p} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Rotation by 35°makes use of a transformation matrix $M_r = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Using $\vec{w}_r = M_r \vec{p} = \begin{bmatrix} 1.89 \\ 2.53 \\ 1 \end{bmatrix}$. So the rotated point $Q = (1.89, 2.53)$. This is not the same physical point as P.

(b) Getting the coordinates of R effectively means rotating P by -35°, although it should be stretched that if you draw this, R and P are actually the same physical point. The coordinates of R turn out to be $(3.03, -0.89)$ in the new coordinate system. Always check these kind of calculating with a sketch: why do you expect R to be in the lower-right quadrant of the rotated system?
(c) The active transformation is the one done in question (a), the passive is question (b).