TUTORIAL 7 - SOLUTIONS

MATRIX RELOADED 2 - VIEWING TRANSFORMATIONS

PART 1: THEORY

EXERCISE 1

(a) What is the viewport transformation? Which parameters characterize it?

The viewport transformation maps the canonical view space to the screen space. It is characterized only by the size of the screen.

(b) What is the orthographic transformation? Which parameters characterize it?

The orthographic transformation allows us to transform the orthographic view space to the canonical view space. It is characterized by the size of the orthographic view volume $[l,r] \times [b,t] \times [f,n]$

(c) In which order do we apply the transformations for the "graphics pipeline"?

Camera, projection, orthographic, viewport.

(d) Why are some of the transformations described by two matrices concatenated?

They can be separated in multiple transformations, for instance translation followed by scaling.

(e) Why can't projective transformations be achieved using affine transformations?

Affine transformations do not allow for a denominator that depends on the coordinates of the vector, which is necessary to account for perspective.

(f) Argue that projective transformations are a more general form of affine transformations.

A projective transformation is one that produces a column vector where the last entry (corresponding to the fictitious dimension) is different from 1.

(g) Are transformations in the "graphics pipeline" active or passive transformations?

Passive, as the object is never moved.

(h) What is the canonical view volume?

It is a cube containing all 3D points with coordinates between -1 and +1, that is $(x,y,z) \in [-1,1]^3$.

PART 2: TRANSFORMATIONS

EXERCISE 2

The following matrix M_{orth} maps the orthographic view volume to the canonical view volume:

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{l+r}{l-r} \\ 0 & \frac{2}{t-b} & 0 & \frac{b+t}{b-t} \\ 0 & 0 & \frac{2}{n-f} & \frac{n+f}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) How is the orthographic view volume defined?
(b) How is the canonical view volume defined?
(c) Where does this transformation map the corners of the orthographic view volume to?
(d) What is the matrix M_{vp} for a screen of size $L_x \times L_y$?
(e) Compute the concatenation $M_{vp}M_{orth}$. Which space does it map to? From which space?

SOLUTIONS

(a) The orthographic view volume is defined as $[l,r] \times [t,b] \times [n,f]]$. So $x \in [l,r]$, $y \in [t,b]$ and $z \in [n,f]$.
(b) The canonical view volume is defined as $[[-1,1] \times [-1,1] \times [-1,1]]$. So $x \in [-1,1]$, $y \in [-1,1]$ and $z \in [-1,1]$.
(c)

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{l+r}{l-r} \\ 0 & \frac{2}{t-b} & 0 & \frac{b+t}{b-t} \\ 0 & 0 & \frac{2}{n-f} & \frac{n+f}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We show that $[l,r] \to [-1,1]$, $[b,t] \to [-1,1]$ and $[n,f] \to [-1,1]$. Remember that the convention is that the camera looks into $-z$ axis, then n has a higher z-value than f.

$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{l+r}{l-r} \\ 0 & \frac{2}{t-b} & 0 & \frac{b+t}{b-t} \\ 0 & 0 & \frac{2}{n-f} & \frac{n+f}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l & r \\ b & t \\ f & n \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{l+r}{l-r} - \frac{2l}{l-r} & \frac{l+r}{l-r} - \frac{2r}{l-r} \\ \frac{b+t}{b-t} - \frac{2b}{b-t} & \frac{b+t}{b-t} - \frac{2t}{b-t} \\ \frac{n+f}{f-n} - \frac{2f}{f-n} & \frac{n+f}{f-n} - \frac{2n}{f-n} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

(d)

$$M_{vp} = \begin{bmatrix} \frac{L_x}{2} & 0 & 0 & \frac{L_x-1}{2} \\ 0 & \frac{L_y}{2} & 0 & \frac{L_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)

$$M_{vp}M_{orth} = \begin{bmatrix} \frac{L_x}{r-l} & 0 & 0 & \frac{L_x-1}{2} + \frac{L_x(l+r)}{2(l-r)} \\ 0 & \frac{L_y}{t-b} & 0 & \frac{L_y-1}{2} + \frac{L_y(b+t)}{2(b-t)} \\ 0 & 0 & \frac{2}{n-f} & \frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It maps from the orthgraphic to the screen space.

EXERCISE 3

Given an eye in position $e = (3,4,2)$, a gaze vector $\vec{g} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and a view-up vector $\vec{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(a) Compute the gaze direction $\hat{g}$.
(b) Write the co-ordinate system given by this information.
(c) Write the rotation matrix needed to map to that co-ordinate system.
(d) Write the translation matrix needed to map to that co-ordinate system from the (x, y, z) co-ordinate system.
(e) Concatenate the two matrices. What do we obtain? Which space does it map to? From which space?

SOLUTIONS

(a) $\hat{g} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(b) $\hat{w} = -\hat{g}$,

$$\hat{u} = \frac{\hat{t} \times \hat{w}}{||\hat{t} \times \hat{w}||} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

and

$$\hat{v} = \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 & 0 \\ \frac{-1}{\sqrt{30}} & \frac{\sqrt{2}}{\sqrt{15}} & \frac{\sqrt{5}}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) We obtain M_{cam}, which maps from the world space to the camera space.

EXERCISE 4

Consider the projection transformation, with the eye gazing into the $-z$ direction. The near plane is at $z = n$ while the far plane is at $z = f$.

(a) Where does it map x and y for a point to?
(b) Where does it map z for a point to? Can it map any value of z?
(c) What is the projection matrix P? Is it an affine transformation?
(d) Apply P to a generic vector $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$. What is the meaning of the last coordinate of the vector?
(e) Write the result of (d) in the extended co-ordinate space (i.e. with the usual extended vector).

SOLUTIONS

(a) It maps x to $\frac{nx}{z}$ and y to $\frac{ny}{z}$.
(b) It maps z to $n + f - \frac{fn}{z}$. No, it can only maps value of z between the near and far plane.

(c)

$$P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It is not an affine transformation because it does not map any vector to one in the form $\left[x, y, z, 1\right]$.

(d) Applying it to P we get $\begin{bmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{bmatrix}$. Every coordinate is then considered to be divided by the last row element.

(e) $\begin{bmatrix} nx/z \\ ny/z \\ n+f-\frac{fn}{z} \\ 1 \end{bmatrix}$.