MIDTERM INFOGR 2016-17
SOLUTIONS

QUESTION 1

Two ways to solve this:

a. We assume that the implicit form equation of the plane is $Ax+By+Cz+D=0$ where $\begin{bmatrix}A\\B\\C\end{bmatrix}$ denotes a vector perpendicular to the plane. We choose $A=1$, $B=2$ and $C=2$, then all we need to do is to determine D, which can be obtained from the fact that $P=(1,1,1)$ is a point on the plane.

$$1\cdot 1+2\cdot 1+2\cdot 1+D=0\rightarrow D=-5$$

So the equation for the plane is $x+2y+2z-5=0$.

b. Consider a point $Q=(x,y,z)$ on the plane. The vectors $\vec{PQ}=\begin{bmatrix}x-1\\y-1\\z-1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}1\\2\\2\end{bmatrix}$ are perpendicular to each other, i.e. $\vec{PQ}\cdot\vec{v}=0$

$$(x-1)\,1+(y-1)\,2+(z-1)\,2=0\rightarrow x+2y+2z-5=0$$

QUESTION 4

The parametric equation of a line starting from $(8,4,0)$ along the vector $\vec{v}=\begin{bmatrix}1\\0\\1\end{bmatrix}$ is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\4\\0\end{bmatrix}+t\begin{bmatrix}1\\0\\1\end{bmatrix}$$

the points of intersection satisfies the equation

$$(x-3)^2+(y-4)^2+z^2=25\rightarrow(t+5)^2+t^2=25\rightarrow t(t+5)=0\rightarrow\begin{cases}t=0\\t=5\end{cases}$$

therefore the intersection points are $P_1=(8,4,0)$ and $P_2=(3,4,-5)$. The center of the sphere is located at $P_0=(3,4,0)$, so by construction the vectors $\vec{OP_1}$ and $\vec{OP_2}$ are the outward normals

$$\vec{OP_1}=\begin{bmatrix}8-3\\4-4\\0-0\end{bmatrix}=\begin{bmatrix}5\\0\\0\end{bmatrix}\quad\vec{OP_2}=\begin{bmatrix}3-3\\4-4\\-5-0\end{bmatrix}=\begin{bmatrix}0\\0\\-5\end{bmatrix}$$

So the outward unit normal vectors are $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\-1\end{bmatrix}$.

QUESTION 6

$\vec{v}\times\vec{w}$ is perpendicular to $\vec{v}$, therefore

$$\vec{v}\cdot(\vec{v}\times\vec{w})=0$$

QUESTION 10

Although this problem looks like a problem in 3 dimensions, note that all points (A,B,P) have zero y-coordinates. So, the triangles $EA'B'$ and EAB are confined to the x-z plane. We use this information to simplify the solution, as we confine the solution below fully to the x-z plane.
The unit vector along $\vec{AB}$ is

$$\frac{1}{\sqrt{(20-8)^2+(12-3)^2}}\begin{bmatrix}20-8\\12-3\end{bmatrix}=\frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix}$$

so the location of P is $(8+\frac{4}{5}l,3+\frac{3}{5}l)$ on the x-z plane. The slope-intercept form equations for $\vec{EA}$ and $\vec{EP}$ are

$$z=\frac{7}{8}x-4\text{ and }z=\frac{7+\frac{3}{5}l}{8+\frac{4}{5}l}x-4$$

respectively. These lines intersect the screen (x-y plane), for which $z = 0$, at

$$x = \frac{32}{7} \text{ (at } A'\text{)} \quad \text{and} \quad x = \frac{32 + \frac{16}{5}l}{7 + \frac{3}{5}l} \text{ (at } P'\text{)}.$$

So,

$$t = \frac{32 + \frac{16}{5}l}{7 + \frac{3}{5}l} - \frac{32}{7} = \frac{\frac{112}{5}l - \frac{96}{5}l}{7\left(7 + \frac{3}{5}l\right)} = \frac{16l}{7(3l + 35)}.$$

Question 11

We have 3 points A, B and C on the plane. Take any two vectors, eg $\vec{A}B = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$ and $\vec{A}C = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$. The cross-product of these two vectors is perpendicular to the triangular plane. The cross product is:

$$\vec{v} = \vec{A}B \times \vec{A}C = \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$$

and the unit vector

$$\hat{v} = \frac{1}{\sqrt{41}} \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$$

So the answer is

$$\pm \frac{1}{\sqrt{41}} \begin{bmatrix} 2 \\ -1 \\ -6 \end{bmatrix}$$

Note: a surface has only two normal vectors differing in sign, so the answer is independent of whether you chose different pairs of edges.

Question 12

This problem is in the same vein as Q10. The unit vector along $\vec{A}B is \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, location of $P = \left(2 + \frac{3}{5}l, 4 + \frac{4}{5}l\right)$. Slope-intercept for one of the straight lines $\vec{E}A$ and $\vec{E}P$ are

$$y = 4x - 4 \quad \text{and} \quad y = \frac{8 + \frac{4}{5}l}{2 + \frac{3}{5}l} x - 4$$

respectively. These lines intersect the x-axis at $x = 1$ (point A') and at $x = \frac{8 + \frac{12}{5}l}{8 + \frac{4}{5}l} = \frac{2 + \frac{3}{5}l}{2 + \frac{l}{5}}$ (point P') respectively. So,

$$t = \frac{2 + \frac{3}{5}l}{2 + \frac{l}{5}} - 1 = \frac{2l}{l + 10}.$$

Question 14

A vector perpendicular to the vector $\vec{A}B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, from which we know that the implicit form looks like

$$-x + y + c = 0$$

since the line passes through $(1, 2)$, $c = -1$ and the equation is

$$x - y + 1 = 0$$

We let $\vec{v}$ pass through $P = (1, 5)$. The parametric equation of such a line is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and it intersects the line $x - y + 1 = 0$ at S, for which

$$(1 - t) - (5 + t) + 1 = 0 \rightarrow t = -\frac{3}{2}$$

Therefore, the coordinates of S are $\left(1 + \frac{3}{2}, 5 - \frac{3}{2}\right) = \left(\frac{5}{2}, \frac{7}{2}\right)$.