Graphics (INFOGR) **Midterm Maths Solutions**

Duration: 1h30m; Total points: 36

No documents allowed. Use of electronic devices, such as calculators, smartphones, smartwatches is forbidden

Multiple choice questions: half of the total points will be deducted for each correct answer not selected, or for each wrong answer selected

Question 1. [2+5=7 points] Consider two points P = $(1, 2, 3)$ and Q = $(5, 10, 11)$ in $\mathbb{R}^3$. They lie on line L.

(a) The general form of the implicit equation of a plane perpendicular to the line L is given by

Your answer: $x + 2y + 2z + D = 0$, with D a constant.

Solution: The unit vector from point P to Q is given by $\hat{u} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.
This means that the general form of the implicit equation of a plane perpendicular to L is given by $x + 2y + 2z + D = 0$, with D a constant. This question is meant as a hint for part (b).

(b) You draw a line from point R = $(3, 8, 5)$ that is perpendicular to line L, intersecting it at point S. The length of the line segment RS is

Your answer: $2\sqrt{2}$.

Solution: The line from point R = $(3, 8, 5)$ that is perpendicular to line L must line on a plane A that is perpendicular to L. The value of D for plane A is then fixed by making sure that the equation $x+2y+2z+D = 0$ holds for point R as well; i.e., $D = -29$.

In order to proceed further, we need to calculate the intersection point of line L and plane A. So we shoot a ray along $\hat{u}$ from P, whose (parametric) equation is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{t}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Using $x+2y+2z-29 = 0$ for plane A in the above equation then yields $t = 6$, so S= $(3, 6, 7)$. The length of the line RS is then $\sqrt{(8-6)^2 + (5-7)^2} = 2\sqrt{2}$.

Question 2. [3+3=6 points] Consider three points A = $(1, 1)$, B = $(-3, 4)$ and C = $(1, 7)$ in $\mathbb{R}^2$.

It is handy to schematically draw these points on a 2D co-ordinate system to see what's going on.

(a) On point C we put a spotlight. What is the length of the shadow of the segment AB on the x-axis?

Your answer: 28/3.

Solution: The unit vector spanning C to A is given by $\hat{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so the loaction of the shadow of A on the x-axis is given by $A_1 = (1, 0)$. Similarly, unit vector spanning C to B is given by $\hat{u}_2 = -\frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. The implicit equation of the ray shot from C along $\hat{u}_2$ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} - \frac{t}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. For the location of the shadow of B on the x-axis ($y = 0!$), given by B_1, we then have $t = 35/3$; i.e., $B_1 = (-25/3, 0)$. So the length of the shadow is $= (1 + 25/3) = 28/3$.

(b) We put a camera on point B, viewing the segment AC, rendering it on the y-axis as the one-dimensional "screen" as A′C′. What is the length of the line segment A′C′?

Your answer: 9/2.

Solution: This part follows the same line as part (a), where we use the equation of the ray shot from C along $\hat{u}_2$ to obtain the location of C′ as $(0, 25/4)$. We then shoot a ray from B along the unit vector towards A, whose parametric form is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \frac{l}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$. For the point where this ray intersects the y-axis we have $l = 15/4$, and the location of A′ is then given by $(0, 7/4)$. So the length of the line segment A′C′ is $= (25/4 - 7/4) = 9/2$.

Question 3. **[1+5+3=9 points]** Given a sphere in $\mathbb{R}^3$ with centre $C = (3, 3, 3)$ and a point on the surface of the sphere $P = (2, 5, 1)$.

(a) The implicit form equation for the sphere is given by

Your answer: $(x - 3)^2 + (y - 3)^2 + (z - 3)^2 = 9$.

Solution: The radius of the sphere is $\sqrt{(3-2)^2 + (3-5)^2 + (3-1)^2} = 3$, so the implicit form equation for the sphere is: $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$.

(b) The location of the point on the surface of the sphere closest to Q = $(6, 9, 1)$ is given by:

Your answer: $(30/7, 39/7, 15/7)$.

Solution: First note that point Q lies outside the sphere.

In order to find the answer we shoot a ray from Q towards the centre of the sphere, and let it intersect the sphere's surface. The parametric equation for this ray, with the unit vector from Q to the centre of the sphere being $\frac{1}{7}\begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix}$, is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} + \frac{t}{7}\begin{bmatrix} -3 \\ -6 \\ 2 \end{bmatrix}$. Substituting this equation in the implicit form equation for the sphere yields the following quadratic equation for t: $t^2 - 14t + 40 = 0$, leading to the solutions $t = 4$ and $t = 10$, so the point we're looking for corresponds to $t = 4$. Use that to obtain the location of the point on the surface of the sphere closest to Q as $(30/7, 39/7, 15/7)$.

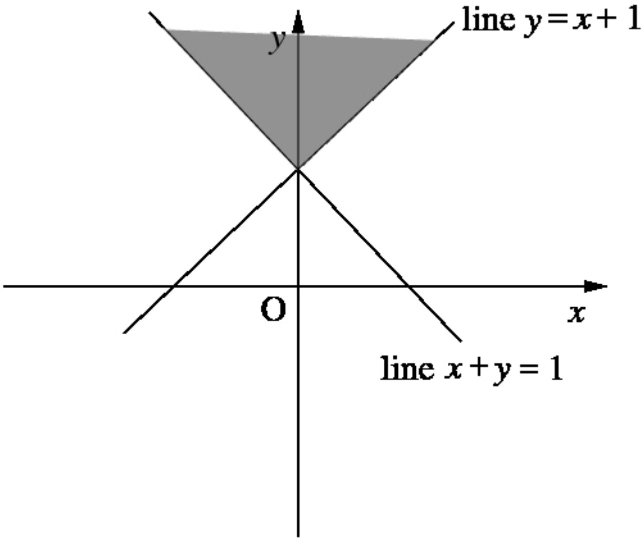
(c) Given $\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is a unit tangent vector of the sphere at point P.

The unit bitangent vector of the sphere at point P is

Your answer: $\begin{bmatrix} \frac{2\sqrt{2}}{3} \\ \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{bmatrix}$.

Solution: The unit vector normal to the surface of the sphere at point P is $\hat{n} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$. So the unit bitangent vector is given by $\hat{n} \times \hat{u} = \begin{bmatrix} \frac{2\sqrt{2}}{3} \\ \frac{1}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{bmatrix}$.

Question 4. [4 points] On a two-dimensional plane there is a co-ordinate system defined (i.e., x- and y-axes and the origin). Shade the region for which **both** conditions $x + y > 1$ and $x + 1 < y$ hold.



Question 5. **[3 points]** The implicit equation of the tangent plane to the sphere $(x-3)^2 + (y-4)^2 + z^2 = 9$ at point P$= (5, 5, 2)$ is given by

Your answer: $2x + y + 2z - 19 = 0$.

Solution: The unit vector normal to the surface of the sphere at point P is $\hat{n} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. So the implicit form equation of the tangent plane passing through P has the general form $2x + y + 2z + D = 0$. The fact that this plane passes through P fixes D as $2 * 5 + 5 + 2 * 2 + D = 0$; i.e., $D = -19$. So the implicit equation of the tangent plane at P is given by $2x + y + 2z - 19 = 0$.

Question 6. **[2+1+4=7 points]** Consider Fig. 1 below, shown in two dimensions. Given:

- The equation of line P is $x - 2y + 1 = 0$.
- The equation of line Q is $y - 2x - 3 = 0$.
- Points A and B lie on line Q. Location of point A is (0,3). The length of the line segment AB is l.
- The points A and B are projected on to line P at A$'$ and B$'$ respectively (i.e., AA$'$ and BB$'$ are both perpendicular to line P).

(a) The length of the line segment AA$'$ is

Your answer: $\sqrt{5}$.

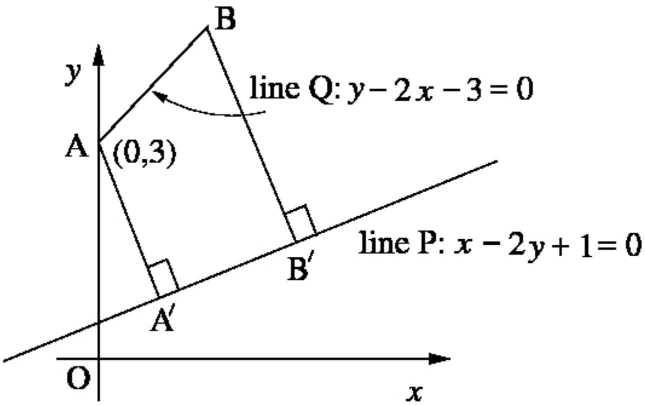


Figure 1: Figure for question 6.

Solution: The unit vector normal to the line P is $\pm\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$. Choose $\hat{u}=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$ and shoot a ray along this unit vector from A. The equation for this ray is $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}+\frac{t}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$.

This ray intersects line P at A′, for which we have $t/\sqrt{5}-2(3-2t/\sqrt{5})+1=0$, i.e., $t=\sqrt{5}$. So the length of the line segment AA′ is $\sqrt{5}$.

(b) The location of the point A′ is

Your answer: $(1,1)$.

Solution: From (a) use $t=\sqrt{5}$ to reach the point $(1,1)$ on line P, which is the location of point A′.

(c) The length of the line segment A′B′ is

Your answer: $\frac{4l}{5}$.

Solution: As the figure shows, the line A′B′ is a projection of the line segment AB on line P. It is therefore obtained by the use of a dot product between the two unit vectors $\hat{v}$ and $\hat{w}$ along AB and A′B′ respectively, since $|\mathrm{A'B'}|=|\mathrm{AB}|(\hat{v}\cdot\hat{w})$. With $\hat{w}\cdot\hat{u}=0$, we have $\hat{w}=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$. In a similar manner, we also have $\hat{v}=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$. So $\hat{v}\cdot\hat{w}=4/5$, which leads to the result that the length of the line segment A′B′ is $4l/5$.