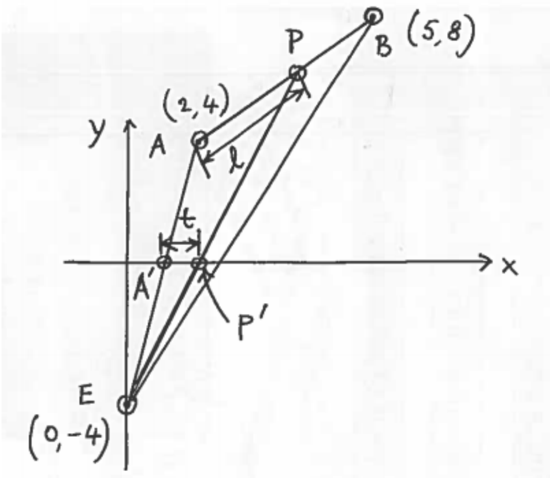
Graphics (INFOGR) **Practice Midterm Solutions**

Duration: 2h; Total points: 50

No documents allowed. Use of electronic devices, such as calculators, smartphones, smartwatches is forbidden

Multiple choice questions: half of the total points will be deducted for each correct answer not selected, or for each wrong answer selected

Question 1. [8 points] As shown in the picture below, a bar AB is placed on the two-dimensional plane, with the locations of A = $(2, 4)$ and B = $(5, 8)$. The bar is being viewed by the eye located at E = $(0, -4)$, and the view is being projected on the one-dimensional "screen", which is simply the x-axis. On the x-axis, A′ is the projection of A, P′ is the projection of P and so on. The distance AP is given by l and the distance A′P′ is given by t.



The quantity t relates to l as $t = \dfrac{2l}{l + 10}$.

Few intermediate steps: (a) first obtain the location of A′, confirm that it is $(1, 0)$, (b) then obtain the same for point P, confirm that it is $(2+3l/5, 4+4l/5)$, (c) finally, obtain the location for point P′, confirm that it is $((2 + 3l/5)/(2 + l/5), 0)$. The rest follows.

Question 2. [4 points] The unit vectors perpendicular to the triangular plane formed by A = $(4, -1, -3)$, B = $(4, 1, -2)$ and C = $(3, -3, -3)$ is given by

Your answer: $\pm\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Calculate the vector AB $= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and AC $= \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$. Use their cross-product, which must be perpendicular to the plane of the triangle, and normalise it.

Question 3. **[3+3=6 points]** Consider the surface of the sphere given by the equation $(x-3)^2 + (y-4)^2 + z^2 = 25$. You shoot a ray from the point $(8, 4, 0)$ along the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The *outward unit* vectors normal to the surface of the sphere at the intersection points of the ray and the sphere are

Your answer: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

Use the parametric form of the ray. Solve for the intersection points, which are $(8, 4, 0)$ itself, and $(3, 4, -5)$. These two points connected to the centre of the sphere are also the normal vectors you're looking for. Normalise them.

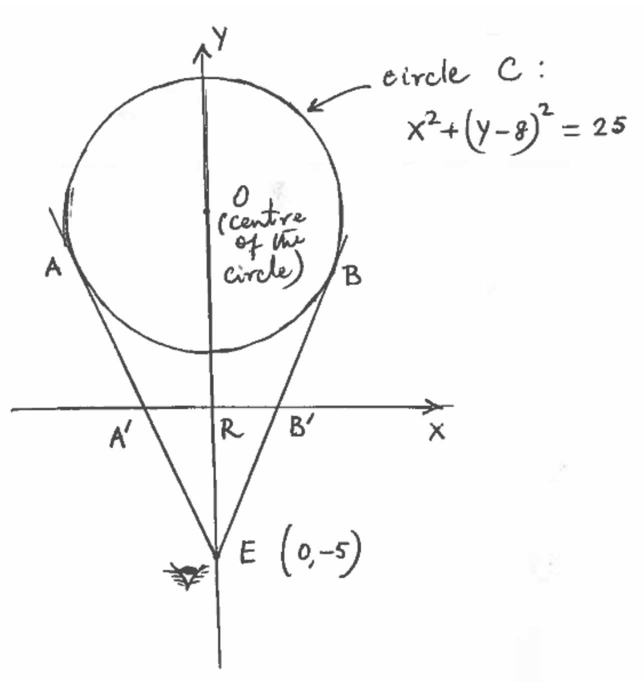


Figure 1: Figure for question 4.

Question 4. **[(1+1)+(5+5)+4=16 points]** Consider Fig. 1 in two dimensions. The equation of circle C is $x^2 + (y-8)^2 = 25$. The eye is located at E $= (0, -5)$. The **maximal** circular arc visible to the eye is AB, which is then being projected on to the one-dimensional "screen" as A′B′.

(a) What are the co-ordinates of a point P on circle C in parametric form?

Your answer: $x = 5\cos\theta$, $y = 8 + 5\sin\theta$.

(b) The co-ordinates of the points A and B are: $\left(\frac{60}{13}, \frac{79}{13}\right)$ and $\left(-\frac{60}{13}, \frac{79}{13}\right)$.

(c) The length of the segment A′B′ is given by $\frac{25}{6}$.

You can either use the parametric form of (a) to find the location of A and B, or just shoot rays from E. Note that EA and EB are tangent rays to the circle, use that to determine the locations of A and B. Note also that the drawing is symmetric on both sides of the y-axis, so obtaining the location of A is enough to get the one for B. The ray shot from E towards B intersects the x-axis at B′, use this info to obtain the location of B′, which you'll use to obtain the length of A′B′.

Question 5. [3+2+4=9 points] Consider the plane $2x + y + 2z - 3 = 0$ in three dimensions, and the point P $= (0, 0, 3)$. Project the point on the plane at Q (meaning that the line PQ is perpendicular to the plane).

(a) The length of the line segment PQ is 1.

(b) The co-ordinates of the point Q is $\left(-\frac{2}{3}, -\frac{1}{3}, \frac{7}{3}\right)$.

(c) The unit vector $\hat{u} = \frac{1}{3}\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ is parallel to the plane. Obtain the second unit vector $\hat{v}$ that is parallel to the plane and perpendicular to $\hat{u}$.

Your answer: $\pm\frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$.

The implicit plane equation immediately yields the unit vector $\hat{w}$ perpendicular to the plane. Use it to obtain the parametric equation of a line perpendicular to the plane. Further, use that to obtain the length of PQ and the location of Q (there were a number of similar exercises in the tutorials). The cross product of $\hat{w}$ and $\hat{u}$ is what you are looking for in part (c).

Question 6. [2 points] There are two vectors

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{an d} \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

The quantity $u = \vec{v} \cdot (\vec{v} \times \vec{w})$ equals 0.

You do not need to explicitly calculate this, since $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$, so the dot product must be zero.

Question 7. **[3 points]** As shown in Fig. 2 below, the angle between the two vectors $\vec{u}$ and $\vec{v}$ is ϕ.

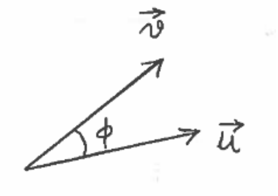


Figure 2: Figure for question 7.

The angle between the vector $\vec{w} = (\vec{u} \times \vec{v}) \times \vec{u}$ and $\vec{v}$ is $90° - \phi$.

Use the "corkscrew" rule (right-handed co-ordinate system) to justify this.