Graphics (INFOGR) **Midterm Maths Solutions**

Duration: 1h30m; Total points: 37

No documents allowed. Use of electronic devices, such as calculators, smartphones, smartwatches is forbidden

Multiple choice questions: half of the total points will be deducted for each correct answer not selected, or for each wrong answer selected

Question 1. **[3+4=7 points]** Consider two points $P = (2, 1)$ and $Q = (-1, 5)$ in $\mathbb{R}^2$.

(a) Write down the equation of the line passing through them in implicit form.

Your answer: $4x + 3y - 11 = 0$.

Solution: The slope of the line is $\frac{5-1}{-1-2} = -\frac{4}{3}$. Equation of the straight line passing through them is therefore $y = -\frac{4}{3}x + c$. Fix $c = \frac{11}{3}$ by the condition that the line passes through P or Q.

(b) The line segment PQ is one arm of a full square PQRS; the vertices are labelled in the clockwise direction. Find the coordinates of R and S.

Your answer: $R = (3, 8)$ and $S = (6, 4)$.

Solution: The length of the line segment PQ is 5. The line segments PS and QR are perpendiclar to PQ, and their lengths are also 5. We'll find the coordinates of R and S using from these, by shooting rays of lengths 5 from P and Q in a direction perpendicular to PQ.

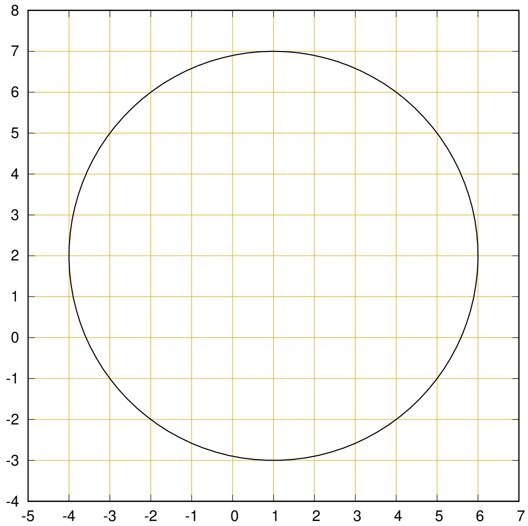
Given that the slope of the line segment PQ is $-\frac{4}{3}$, the parametric form equation of a ray shot from (x_0, y_0) in a direction perpendicular to PQ is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{l}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. To find the location of R, we use $(x_0, y_0) = (-1, 5)$ and $l = 5$, leading to $(3, 8)$. Similarly, to find the location of S, we use $(x_0, y_0) = (2, 1)$ and $l = 5$, leading to $(6, 4)$.

Question 2. **[2+4=6 points]** Consider the circle in part (b) of this question. The centre of the circle is located at (1,2).

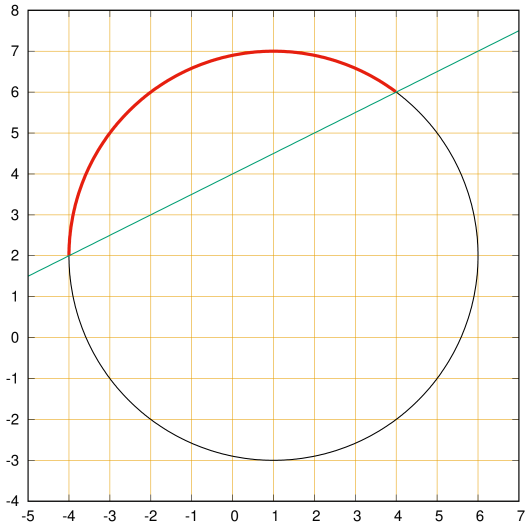
(a) Write down the equation of this circle.

Your answer: $(x - 1)^2 + (y - 2)^2 = 25$.

(b) Identify clearly the two points (let's call them A and B) on the circle below such that on the arc AB the condition $x - 2y + 8 \leq 0$ holds; shade this arc.



Your answer:



Equation of the green line is $x - 2y + 8 = 0$. This line intersects the circle at $(4, 6)$ and $(-4, 2)$; these are the coordinates of A and B (irrespective of order). All points above the green line (and including the green line) satisfy the condition $x - 2y + 8 \leq 0$. The arc of the circle that satisfies this condition is shaded red.

Question 3. **[3 points]** Consider two points $P = (3, 3, 3)$ and $Q = (5, 4, 1)$ on the plane $x + 2y + 2z = 15$ in $\mathbb{R}^3$. Calculate the unit vector $\hat{v}$ perpendicular to PQ and parallel to the plane.

Your answer: $\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$.

Solution: The unit vector normal to the plane is given by $\hat{n} = \pm\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. The unit vector along PQ is $\hat{u} = \pm\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. The answer is given by $\hat{v} = \hat{n} \times \hat{u}$.

Question 4. **[3+3=6 points]** Consider the point $L = (7, 4, 7)$ in $\mathbb{R}^3$ at which a light source is placed. Consider also a bar with ends located at $P = (5, 3, 5)$ and $Q = (1, 2, 4)$, casting a shadow P_1Q_1 on the $z = 1$ plane. Find the coordinates of P_1 and Q_1.

Your answer: $P_1 = (1, 1, 1)$ and $Q_1 = (-5, 0, 1)$.

Solution: Shoot a ray from the light source L towards P. The equation of this ray is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix} - \frac{l}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, where $\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ is the unit vector from L towards P. The intersection point of this line with the $z = 1$ plane corresponds to $l = 9$, and the intersection point is given by $P_1 = (1, 1, 1)$.

Similarly, shoot a ray from the light source L towards Q. The equation of this ray is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix} - \frac{t}{7}\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$, where $\frac{1}{7}\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$ is the unit vector from L towards Q. The intersection point of this line with the $z = 1$ plane corresponds to $t = 14$, and the intersection point is given by $Q_1 = (-5, 0, 1)$.

Question 5. **[4+3=7 points]** Consider the plane $2x+3y+6z = 8$ and a point $P = (5, 6, 13)$ in $\mathbb{R}^3$.

(a) Obtain the minimum distance between the point P and the plane.

Your answer: 14.

Solution: To calculate the minimum distance we simply need to shoot a ray normal to the plane from P, and let it intersect the plane at Q. Then the length of PQ is the minimum distance we seek.

The equation of this ray is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 13 \end{bmatrix} + \frac{t}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$, where $\frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ is the unit normal vector to the plane. The intersection point of this ray and the plane is obtained from the solution $2(5 + 2t/7) + 3(6 + 3t/7) + 6(13 + 6t/7) = 8$, i.e., $t = -14$. So the minimum distance is $|t| = 14$.

(b) Find the coordinates of the point Q on the plane corresponding to the minimum distance in part (a).

Your answer: $(1, 0, 1)$.

Use $t = -14$ in the equation of the ray.

Question 6. **[2+4+2=8 points]** Consider the sphere $x^2 + y^2 + z^2 - 6x - 6y - 6z + 18 = 0$ in $\mathbb{R}^3$.

(a) What is the centre and the radius of this circle?

Your answer: $(3, 3, 3)$ and 3.

Solution: The equation of the circle can be written as $(x-3)^2 + (y-3)^2 + (z-3)^2 = 9$.

(b) Consider the point C $= (3, -1, 4)$, where a camera is located. A ray is shot in the direction $\hat{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Find the coordinates of the points where this ray intersects the sphere. Which one(s) is/are visible to the camera?

Your answer: $(3, 0, 3)$ and $(3, 3, 0)$. Only the first one is visible to the camera.

The equation of this ray is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + \frac{t}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$; i.e., $x = 3$, $y = -1 + \frac{t}{\sqrt{2}}$, $z = 4 - \frac{t}{\sqrt{2}}$. Using these in the equation of the circle yields $(-4 + t/\sqrt{2})^2 + (1 - t/\sqrt{2})^2 = 9$ or $t^2 - 5\sqrt{2}t + 8 = 0$, whose solutions are

$t = \sqrt{2}$ and $t = 4\sqrt{2}$. The intersection points are obtained by using these values in the equation of the ray.

(c) Write down the equation of the plane tangent to the surface of the sphere at the point that is not visible to the camera.

Your answer: $z = 0$.

The unit vector from the centre of the circle to the point $(3, 3, 0)$i, which is the one not visible to the camera, is given by $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, so the equation of the plane must be $z =$ constant, and the constant is then determined to be 0 since the plane passes through $(3, 3, 0)$.