Graphics (INFOGR) **Final Maths Practice Solutions**

Duration: 1h30m; Total points: 43

No documents allowed. Use of electronic devices, such as calculators, smartphones, smartwatches is forbidden

Question 1. **[5 points]** Consider three points P = $(2, 2, 2)$, Q = $(5, 4, 5)$ and R = $(3, 5, 4)$ in $\mathbb{R}^3$. Find the implicit equation of a plane passing through these points.

Your answer: $5x + 3y - 7z - 2 = 0$.

Solution: The vector spanning from P to Q is $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$, and the vector spanning from P to R is $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. The cross product of these two vectors is $\vec{u} \times \vec{v} = \begin{bmatrix} -5 \\ -3 \\ 7 \end{bmatrix}$. So the equation of the plane must be of the form $5x + 3y - 7z + D = 0$. The fact that all three points lies on this plane (any one will do) then leads to $D = -2$.

Question 2. **[(2+1+2)+5+2+2+2=16 points]** Consider the point P = $(6, 4, 6)$ and the plane L: $6x + 3y + 2z - 11 = 0$ in $\mathbb{R}^3$, measured in co-ordinate system 1.

It can be handy to schematically draw these to see what's going on.

(a) We translate the origin of the co-ordinate system to point $(1, 1, 1)$ of co-ordinate system 1 to define a (new) co-ordinate system 2. Write down the transformation matrix M_t that transforms the co-ordinates of the point X, which is (x, y, z) in co-ordinate system 1 to (x', y', z') in co-ordinate system 2. Apply this transformation to obtain the location of point P as well as the equation of the plane L in the co-ordinate system 2.

Your answer: $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$. P = $(5, 3, 5)$ and L: $6x' + 3y' + 2z' = 0$ in co-ordinate system 2.

(b) Write down the transformation matrix M_p that projects the point P on to the plane L in co-ordinate system 2. How would you characterise such a matrix?

Your answer: $M_p = \begin{bmatrix} 13/49 & -18/49 & -12/49 & 0 \\ -18/49 & 40/49 & -6/49 & 0 \\ -12/49 & -6/49 & 45/49 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is a symmetric matrix.

Solution: The normalised vector to the plane is given by $\hat{n} = \frac{1}{7}\begin{bmatrix} 6 \\ 3 \\ 2 \\ 0 \end{bmatrix}$, which leads to the transformation matrix above, since the plane passes through the origin of coordinate system 2.

(c) The point P projected on the plane is denoted by P′. Calculate P′ in co-ordinate system 2.

Your answer: $(-1, 0, 3)$.

Solution: Apply the projection matrix on the location of P in co-ordinate system 2.

(d) Now tranaslate the origin of co-ordinate system 2 back to the location of the origin in co-ordinate system 1. Write down the transformation matrix M_t' that transforms the co-ordinates of the point X, which is (x', y', z') in co-ordinate system 2 to (x, y, z) in the co-ordinate system 1. Obtain the location of the projected point P′ in co-ordinate system 1.

Your answer: $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t'} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$. $\mathrm{P}' = (0, 1, 4)$.

(e) In co-ordinate system 1 if you were to obtain the location of P′ directly from P using a transformation matrix M, then write down the expression of M in terms of M_t, M_p and M_t'.

Your answer: $M = M_t' M_p M_t$.

Question 3. **[3+3=6 points]** Consider again the point $\mathrm{P} = (6, 4, 6)$ and the plane L: $6x + 3y + 2z - 11 = 0$ in $\mathbb{R}^3$.

(a) Use the method of shooting a ray from point P to project it on plane L, and obtain the location of the projected point P′.

Your answer: $(0, 1, 4)$.

Solution: Shoot a ray from P $= (6, 4, 6)$ in the direction $\hat{n}$. The equation of the ray is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix} + t\hat{n}$. Using this on the plane L yields the solution $t = -7$, and further, the location of the projected point to be $(0, 1, 4)$. This actually serves as a simple check to question 2.

(b) Reflect the point P on the plane L, and obtain the location of the reflected point Q.

Your answer: $(-6, -2, 2)$.

Solution: This is an extension of part (a). All we need to do is to double the value of t as obtained in part (a).

Question 4. **[3+3=6 points]** Consider the line $2x - y + 3 = 0$ in $\mathbf{R}^2$.

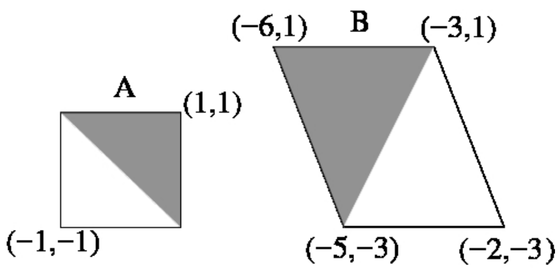
(a) The 2×2 matrix that describes the reflection of a vector on the line is given by

Your answer: $\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$.

(b) The unit vector that remains unchanged when reflected on this line is given by

Your answer: $\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Question 5. **[4+6=10 points]** You need to transform the square A to the rhombus B in the figure below (figure not to scale).



(a) What are the elementary active transformations to you need to achieve this?

Your answer: reflection about y-axis, scaling, followed by $-x$-shearing and then translation. (The first two are diagonal matrices, so their ordering does not matter).

(b) Write down the required transformation matrix.

Your answer: $\begin{bmatrix} -3/2 & -1/2 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$= \underbrace{\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{M_t} \underbrace{\begin{bmatrix} 1 & -1/4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{sh}} \underbrace{\begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{sc}} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{ref}}.$$