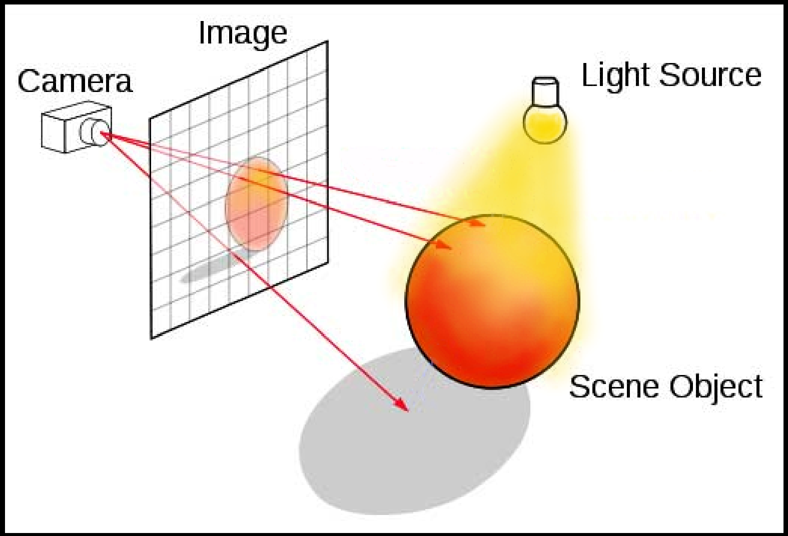
Graphics (INFOGR), 2018-19, Block IV, lecture 1

Deb Panja

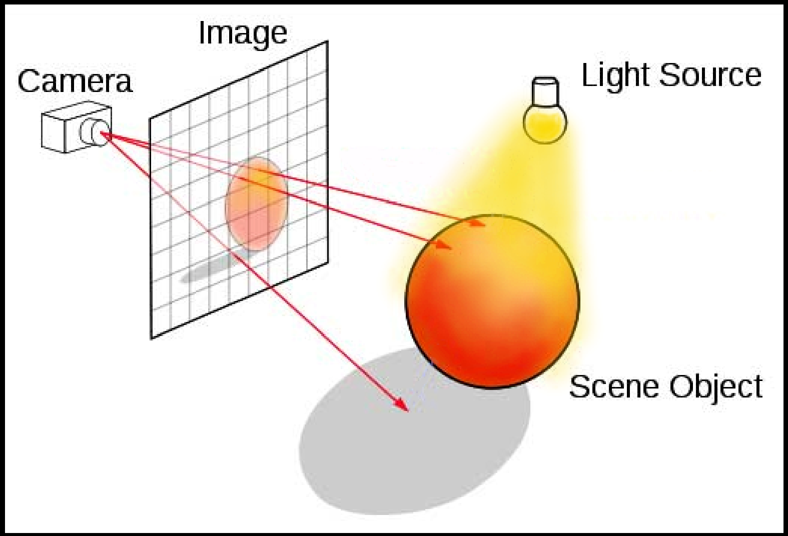
Today: Vectors and vector algebra

Welcome

Ray tracing – part I of the course



Ray tracing – part I of the course



- Why vectors?
  - you need to shoot a lot of such rays!
  - vectors are the vehicles you need for ray tracing

And vectors as vehicles for....

- Define a virtual scene
- Define a camera direction
- Trace a bullet around a scene
- Line-of-sight queries
- And greetings from the gaming community (NVIDIA, Microsoft)....
  - https://blogs.nvidia.com/blog/2018/03/21/ (epic-games-reflections-ray-tracing-offers-peek-gdc)
  - https://www.youtube.com/watch?v=-zW3Ghz-WQw
  - https://www.youtube.com/watch?v=81E9yVU-KB8

Scalars (before we talk about vectors!)

- Quantities that can be described by a magnitude (i.e., a single number)

Scalars

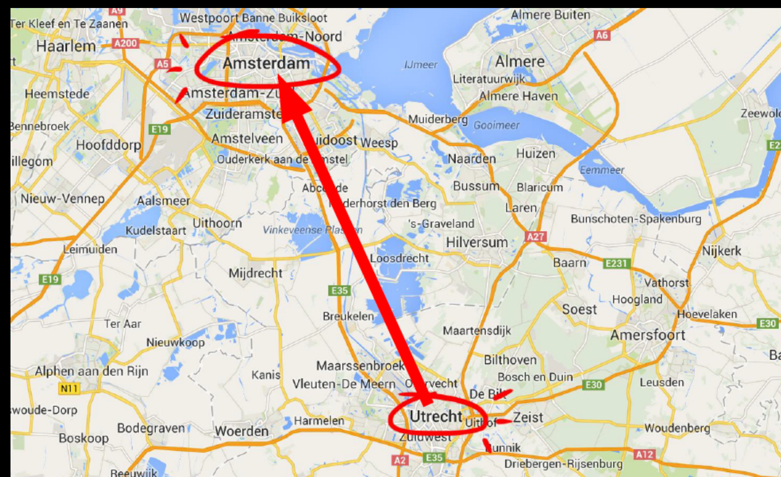
- Quantities that can be described by a magnitude (i.e., a single number)
  - this sack of potatoes weighs 5 kilos
  - distance between Utrecht and Amsterdam is 40.5 kms
  - the car is travelling with speed 50 km/h
  - numbers like $\pi = 3.14159\ldots$, $e = 2.71818\ldots$, $1/3$, $-1/\sqrt{2}$ etc.
- On a computer: int, float, double

Vectors

- Quantities that have not only a magnitude but also a direction

Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)



  - the velocity of an airplane

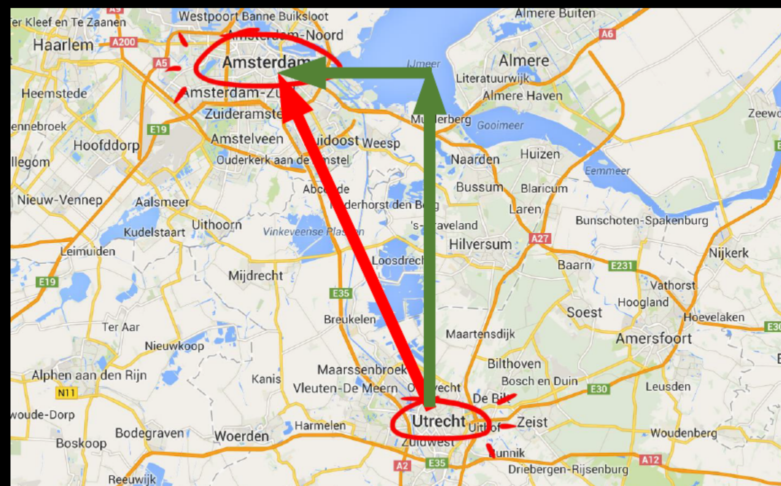
Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)

- One way to represent the U-A vector:
  - start at U and end at A; vector (the arrow!) spans the two
  - start-point (U): move 40.5 kms in 24° west of north

Vectors

- Quantities that have not only a magnitude but also a direction
  - Utrecht-Amsterdam example (40.5 kms in-between)



- Equivalent second way to represent the U-A vector:
  - start-point (U): move 37 kms north and 16.47 kms west
    (“north” and “west” are reference directions)

Reference directions $\Rightarrow$ a co-ordinate system

- Number of reference directions = dimensionality of space
  d-dimensional space $\equiv \mathbb{R}^d$; 2D $\equiv \mathbb{R}^2$, 3D $\equiv \mathbb{R}^3 \ldots$

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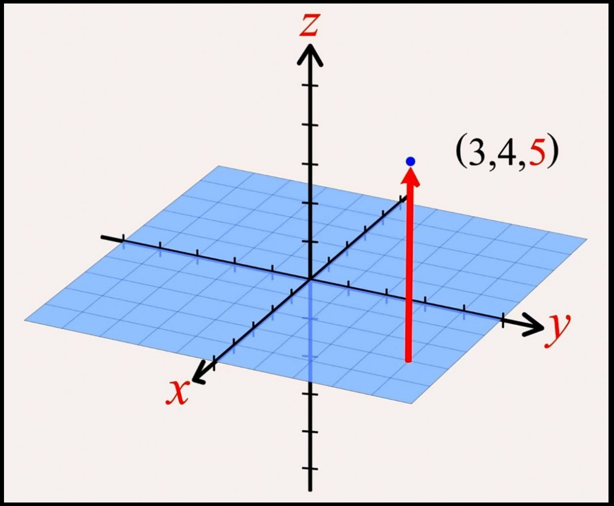
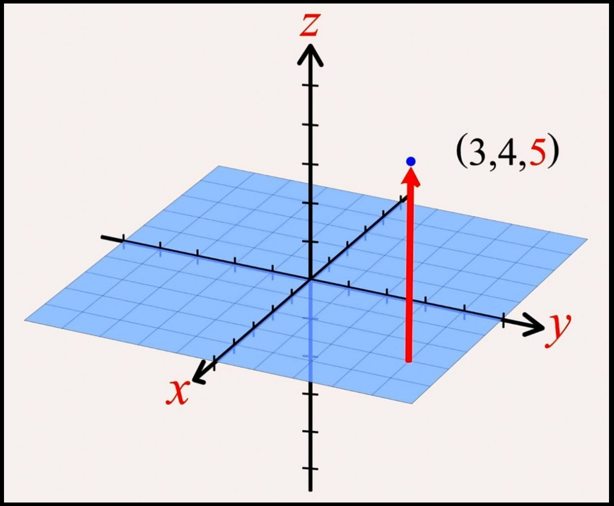
- Cartesian co-ordinate system
  in 3D:
  (reference directions $\perp$ to each other)



Reference directions ⇒ a co-ordinate system

- Cartesian co-ordinate system in 3D:



- A point P is represented
  - by (x, y) co-ordinates in two dimensions
  - by (x, y, z) co-ordinates in three-dimensions
  - by $(x_1, x_2, \ldots, x_d)$ co-ordinates in d dimensions
- Origin of a co-ordinate system: all entries of P are zero

Reference directions $\Rightarrow$ a co-ordinate system

- Co-ordinate system does not have to be orthogonal/Cartesian!

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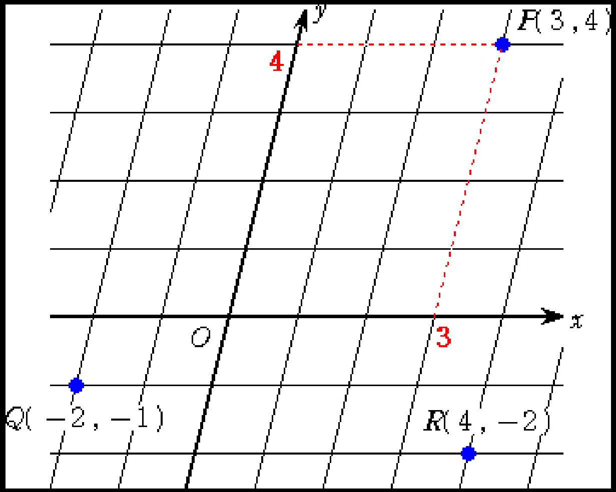
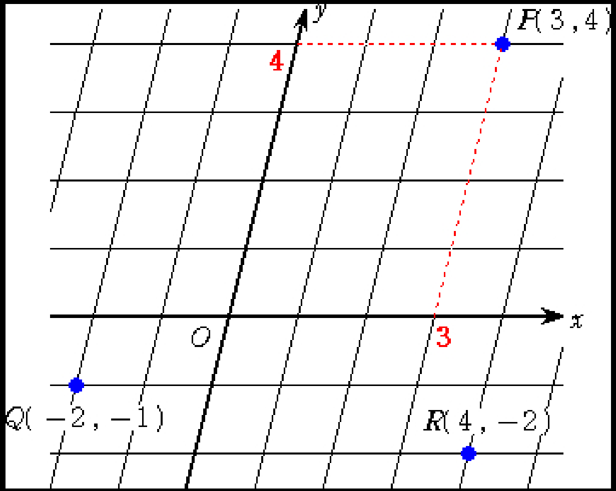


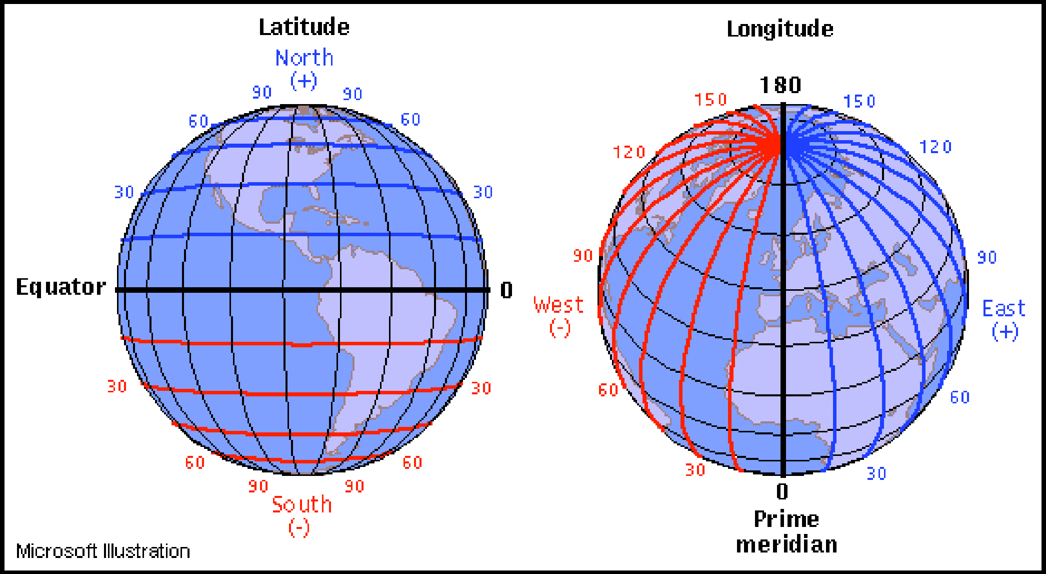
Reference directions ⇒ a co-ordinate system

- Co-ordinate system does not have to be orthogonal/Cartesian!

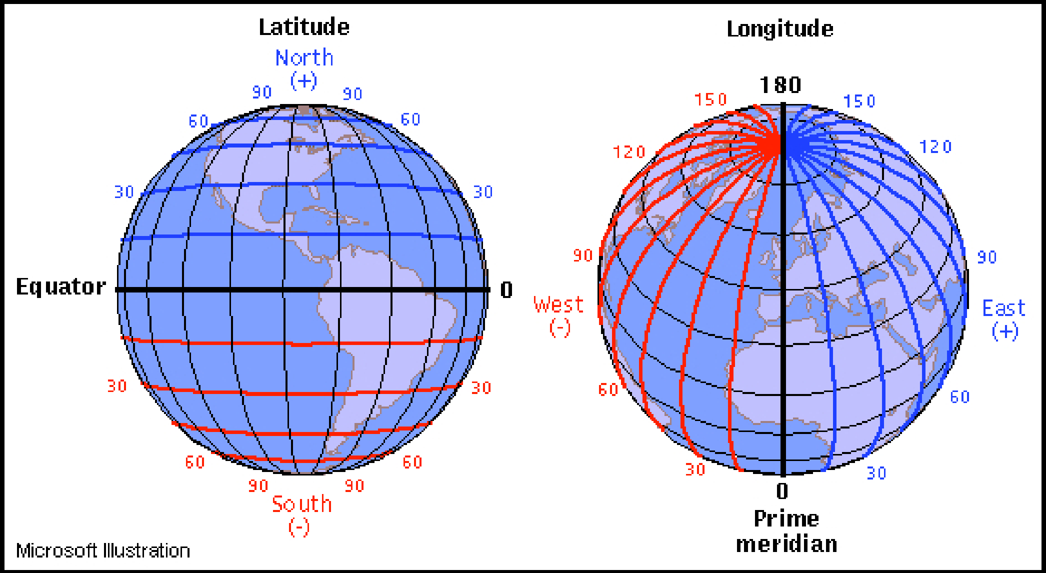


- There are advantages for Cartesian co-ordinate syetems (later)

Q. Latitude-longitude: is it a co-ordinate system?



A. Latitude-longitude: It is a co-ordinate system



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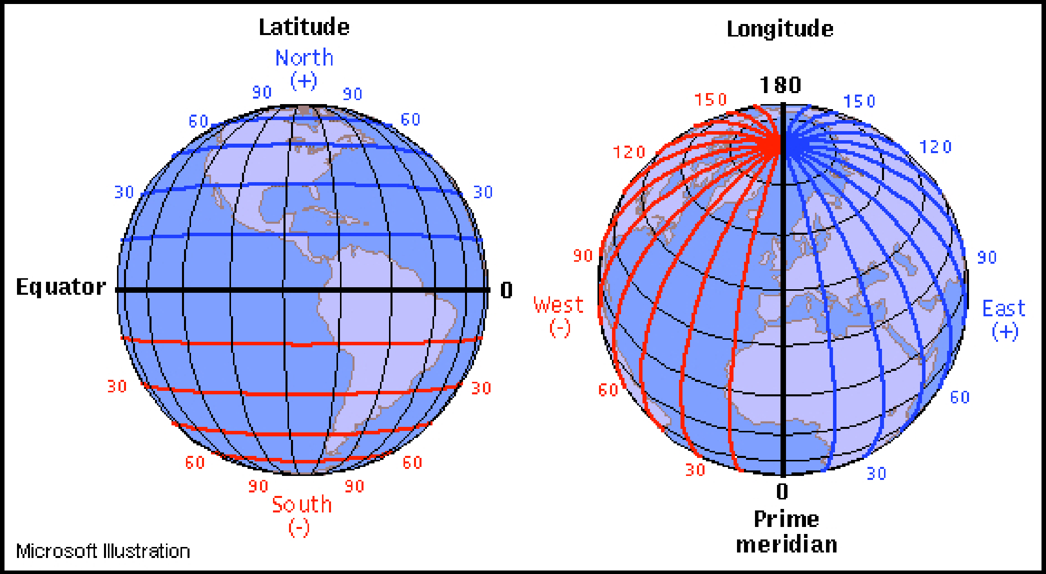
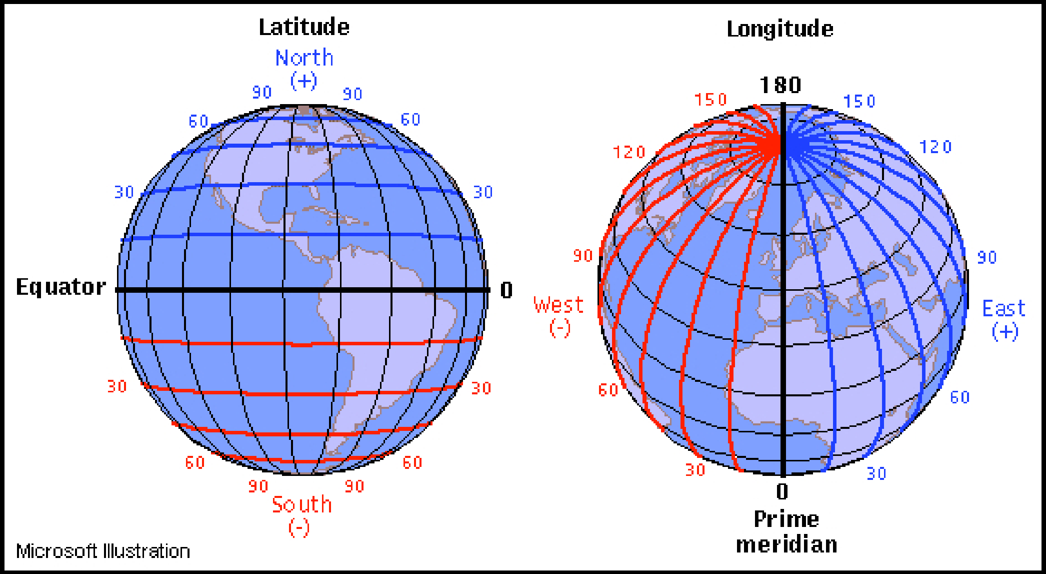


Q. Is it orthogonal?

Latitude-longitude: It is a co-ordinate system



A. It is (locally) orthogonal

A point in a co-ordinate system

- Is represented as an array on a computer

A point in a co-ordinate system

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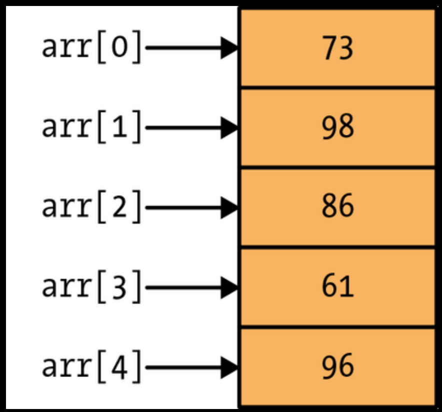
  - example in 5 dimensions ($d = 5$): $P = (73, 98, 86, 61, 96)$



A vector in a co-ordinate system

- Like Utrecht $\rightarrow$ Amsterdam (37 kms N, 16.47 kms W)

A vector in a co-ordinate system

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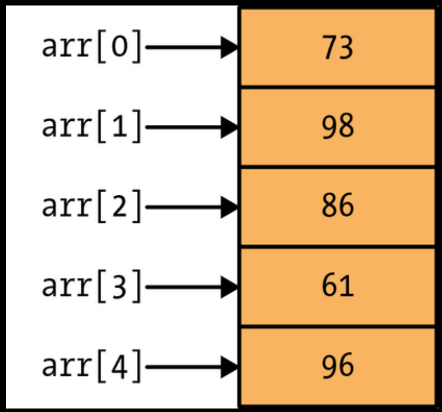
  - an example vector in 5 dimensions ($d = 5$): $\vec{v} = (73, 98, 86, 61, 96)$ is *also* represented as an array on the computer!



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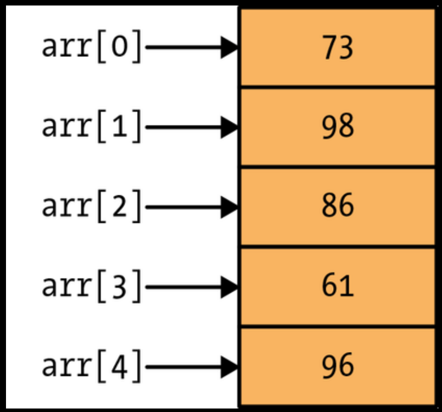
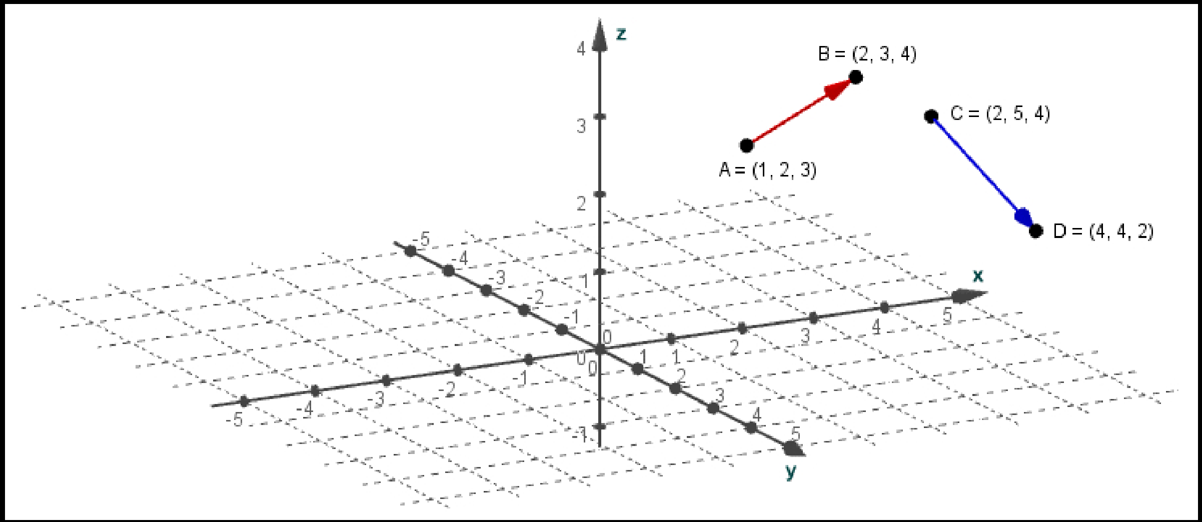
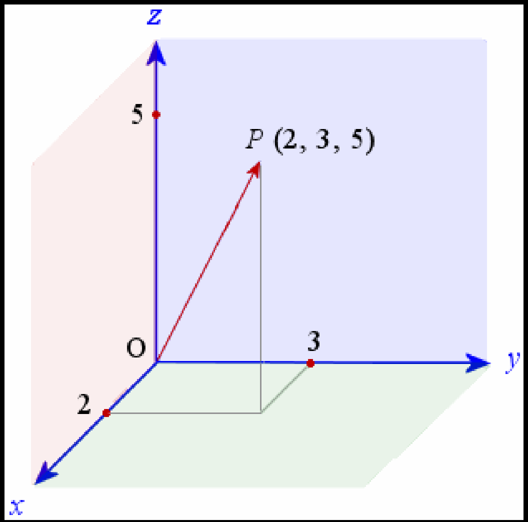


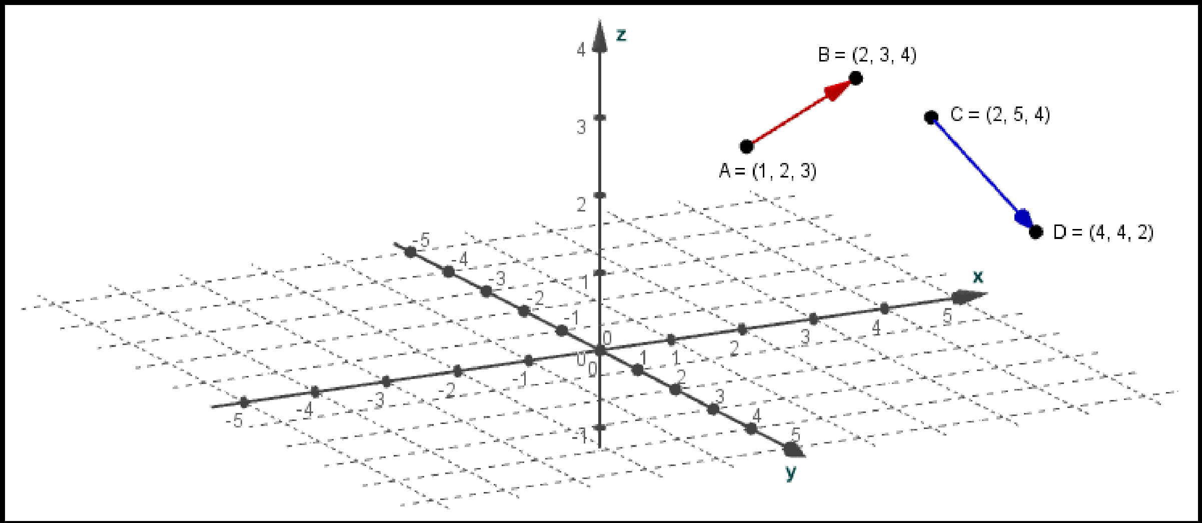
- So... what is the difference between a point and a vector?

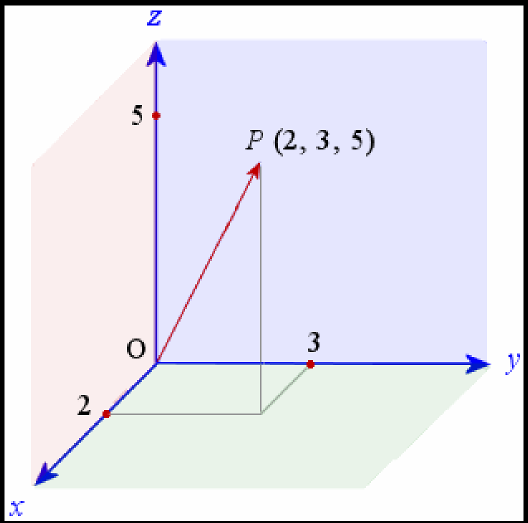
A point (on a co-ordinate system) vs a vector (e.g., in 3D)





A point (on a co-ordinate system) vs a vector (e.g., in 3D)





- A vector does not specify the starting point!
  - it only specifies the length and the direction of the arrow

Summary so far...

- Scalar: Quantity represented by a magnitude (a single number)
- Vector: Quantity requiring a magnitude and a direction
  - to represent it, we need a co-ordinate system

    (a) does not have to be Cartesian

    (b) we will use Cartesian unless otherwise stated
  - number of reference directions = number of spatial dimensions
  - A point P is represented by $(x_1, x_2, \ldots, x_d)$ in d spatial dimensions [by (x, y) in 2D and by (x, y, z) in 3D]
  - both points and vectors are represented by an array on a computer (a vector is however fundamentally different entity than a point)

Point and vector representation

- Point P: $(x_1, x_2, \ldots, x_d)$ in d spatial dimensions
  - by (x, y) in 2D and by (x, y, z) in 3D
- Vector $\vec{v}$: $\begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$ in d spatial dimensions: vector notation
  - by $\begin{bmatrix} v_x \\ v_y \end{bmatrix}$ in 2D and by $\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$ in 3D
  - the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ spans the origin and the point (x, y, z) in 3D

Vector addition

(Only for vectors of the same dimension!)

- Vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$

- Vector $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ . \\ . \\ u_d + v_d \end{bmatrix}$

Vector subtraction

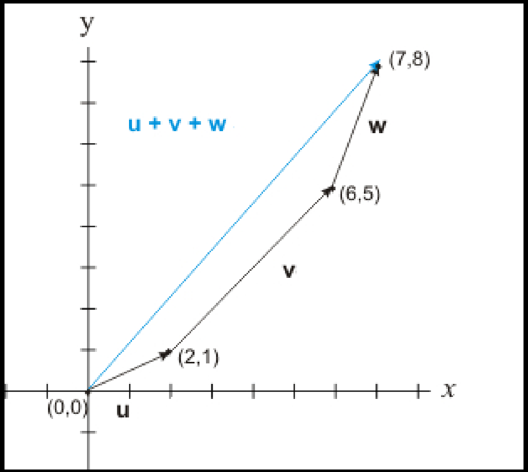
(Only for vectors of the same dimension!)

- Vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$
- Vector $\vec{u} - \vec{v} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ . \\ . \\ u_d - v_d \end{bmatrix}$
- $\vec{u} - \vec{v} = 0 \quad \Rightarrow \quad \vec{u} = \vec{v} \quad \Rightarrow \quad u_1 = v_1, u_2 = v_2, \ldots, u_d = v_d$

Vector addition and subtraction: example

- Addition: $\vec{a} = \vec{u} + \vec{v} + \vec{w}$

Example: $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

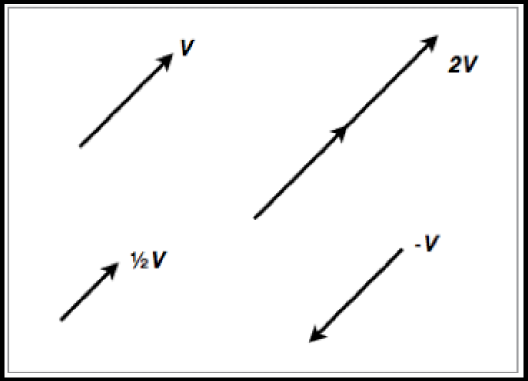


- Subtraction: $\vec{a} - \vec{w}$; reverse the direction of $\vec{w}$ and vector-add to $\vec{a}$ (i.e.,to get $-\vec{w}$ simply reverse the arrow)

Scalar multiplication of a vector

- Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$, scalar λ

$$\lambda\vec{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ . \\ . \\ \lambda v_d \end{bmatrix}$$



Magnitude (length, or norm) of a vector

(Formulas below holds for Cartesian co-ordinate system only!)

- Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$; magnitude (norm) $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2}$

  2D: $||\vec{v}|| = \sqrt{v_x^2 + v_y^2}$, 3D: $||\vec{v}|| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

  ($||\vec{v}||$ is the length of the arrow)

Magnitude (length/norm) of a vector, and unit vector

(Formulas below holds for Cartesian co-ordinate system only!)

- Vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$; magnitude (norm) $||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_d^2}$

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  ($||\vec{v}||$ is the length of the arrow)

- Corresponding unit vector $\hat{v} = \dfrac{1}{||\vec{v}||} \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$; you can confirm $||\hat{v}|| = 1$

  (this process is called normalisation)

Magnitude (length) of a vector, unit and basis vectors

(Formulas below holds for Cartesian co-ordinate system only!)

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  – unit vectors in reference directions $\hat{x}_1, \hat{x}_2, \ldots$ are the basis vectors
    (e.g., $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the basis vectors in 3D; in vector notation?)

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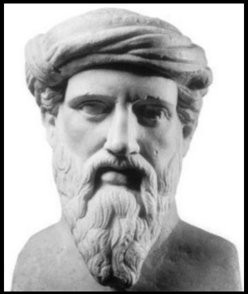
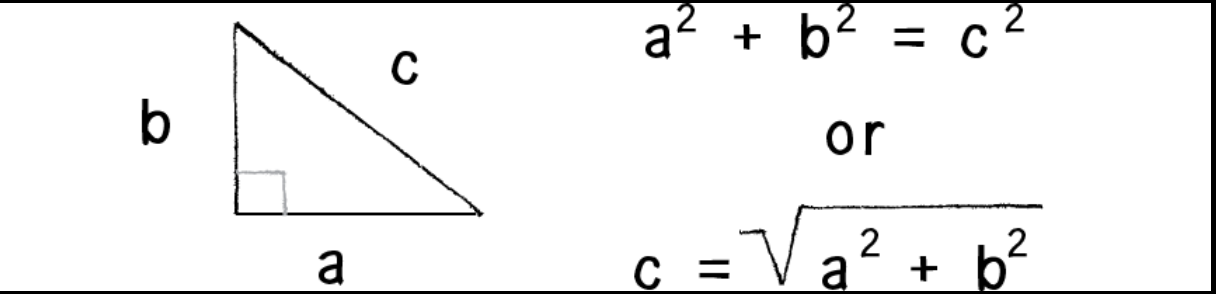
- Why do we need a Cartesian co-ordinate system?

Pythagoras' theorem and elementary trigometry
(works for Cartesian co-ordinate system only!)

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$ (Pythagoras)

Pythagoras' theorem





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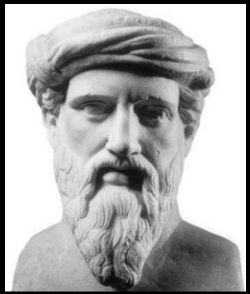
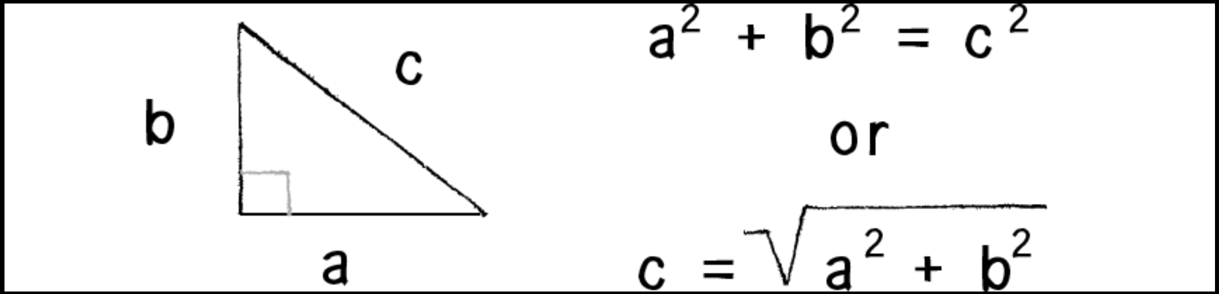
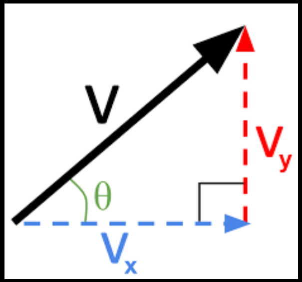




(below it works for Cartesian co-ordinate system only!)

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$

  e.g., $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $||\vec{v}|| = \sqrt{3^2 + 4^2} = 5$



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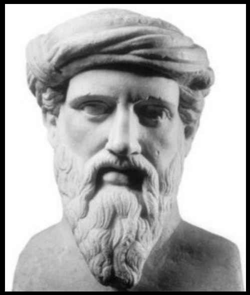

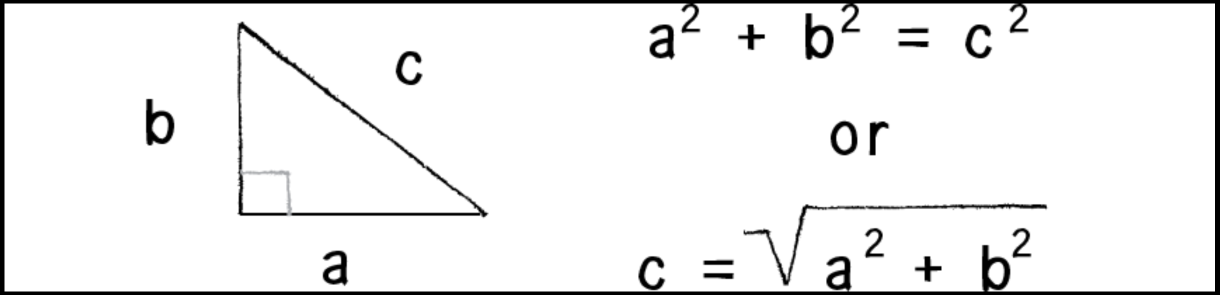

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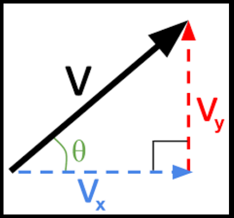

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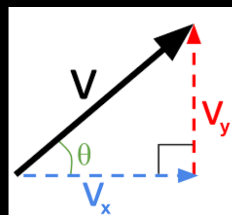
- Pythagoras in d dimensions: $||\vec{v}||^2 = v_1^2 + v_2^2 + \ldots + v_d^2$

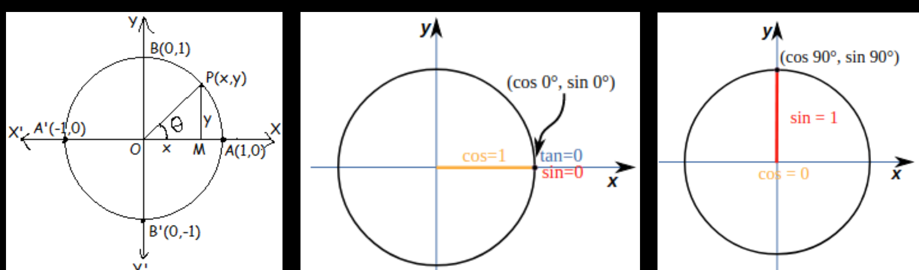
- Null vector: vector of magnitude zero; $v_1 = v_2 = \ldots = v_d = 0$

Cartesian co-ordinate syetem vectors and trigonometry

- In 2D: basis vectors $\hat{x}, \hat{y}$; $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$; $||\vec{v}||^2 = v_x^2 + v_y^2$ (Pythagoras)



- $\cos\theta = \dfrac{v_x}{||\vec{v}||}, \sin\theta = \dfrac{v_y}{||\vec{v}||}, \tan\theta = \dfrac{v_y}{v_x}$



unit circle

Exact Values of Trigonometric Functions

Angle (θ) Degrees	Angle (θ) Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	Not Defined

Cartesian co-ordinate syetem vectors and trigonometry

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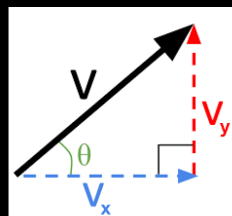

- $\cos\theta = \dfrac{v_x}{||\vec{v}||}$, $\sin\theta = \dfrac{v_y}{||\vec{v}||}$, $\tan\theta = \dfrac{v_y}{v_x}$
- $v_x^2 + v_y^2 = ||\vec{v}||^2 \quad \Rightarrow \quad \sin^2\theta + \cos^2\theta = 1$

Q. What is $\hat{v}$ in terms of θ?
(think in terms of the unit circle!)

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Angle (θ)		sin(θ)	cos(θ)	tan(θ)
Degrees	Radians			
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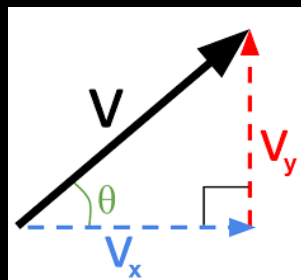

- $\cos\theta = \dfrac{v_x}{||\vec{v}||}, \sin\theta = \dfrac{v_y}{||\vec{v}||}, \tan\theta = \dfrac{v_y}{v_x}$
- $v_x^2 + v_y^2 = ||\vec{v}||^2 \quad \Rightarrow \quad \sin^2\theta + \cos^2\theta = 1$

A. $\hat{v} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

Exact Values of Trigonometric Functions

Angle (θ)		$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
Degrees	Radians			
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	Not Defined

The second summary...

- Vector operations
  - addition and subtraction (requires same dimensionality)
  - scalar multiplication
  - magnitude/length/norm of a vector; unit, null and basis vectors
  - Pythagoras theorem
  - elementary trigonometry: definitions of $\sin, \cos, \tan$

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- Next class: vector algebra (contd.), and shooting rays to objects in 2D

Finally, references

- Book chapter 2: Miscellaneous Math
  - Sec. 2.3
  - Secs. 2.4.1-2.4.2, 2.4.5