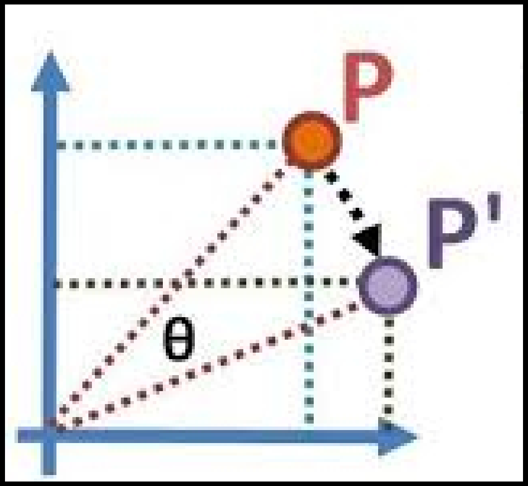
Graphics (INFOGR), 2018-19, Block IV, lecture 11

Deb Panja

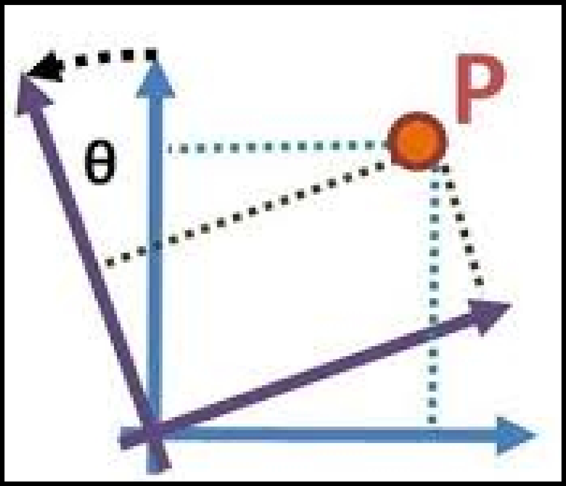
Today: Matrix reloaded 1 (aka Transformations)

Welcome

Recall: active vs. passive transformations



point transformation
(aka active transformation)



co-ordinate transformation
(aka passive transformation)

Today

- Point, or active transformations (using matrices)
  - translation (redo)
  - projection
  - reflection
  - scaling
  - shearing
  - rotation
  - linear transformation properties
  - combining transformations (and transformation back!)
- Co-ordinate, or passive transformations
- Will largely work in terms of symbols

Point, or active transformations

Translation

Translation as a matrix operation

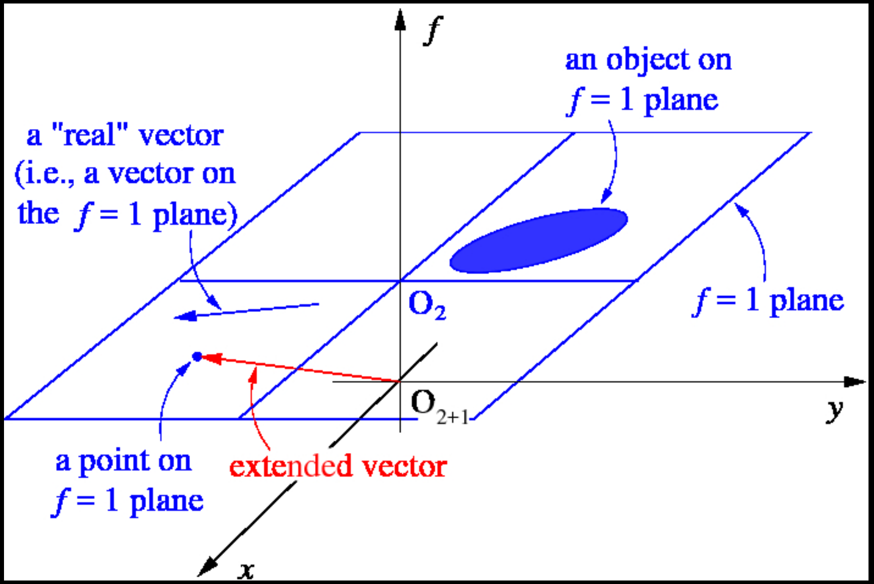
- We translate a point P (x, y, z) by (a_x, a_y, a_z)
  i.e., $x' = x + a_x$, $y' = y + a_y$, $z' = z + a_z$

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}}_{\vec{w}_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t(\vec{a})} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\substack{\text{extended} \\ \text{vector}}} ; \; \vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

- From now on, will use the extended vector to reach P from the origin by adding a fictitious dimension, meaning:

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; \hat{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

How to think about an extended vector in 2D

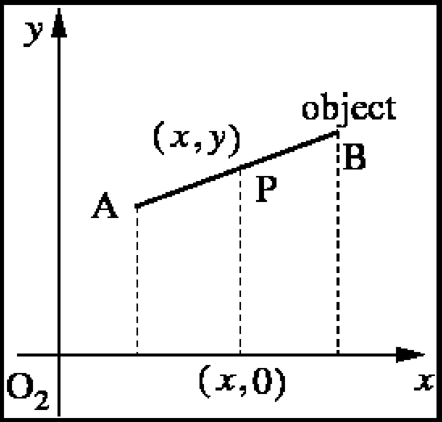


- Note: A "real" vector $\vec{v}$, by construction, satisfies $\vec{v} \cdot \hat{f} = 0$

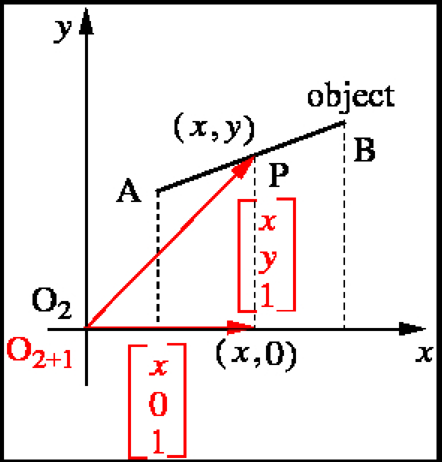
  e.g., the (2+1)D representation of a real vector in 2D is $\begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$

Projection

Projecting a 2D object on the x-axis: matrix operation?

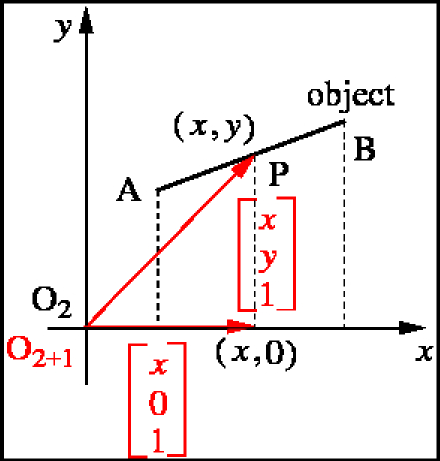


Projecting a 2D object on the x-axis: a matrix operation



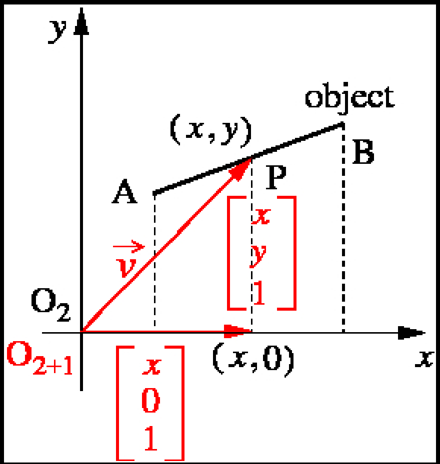
$$\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projecting a 2D object on the x-axis: a matrix operation



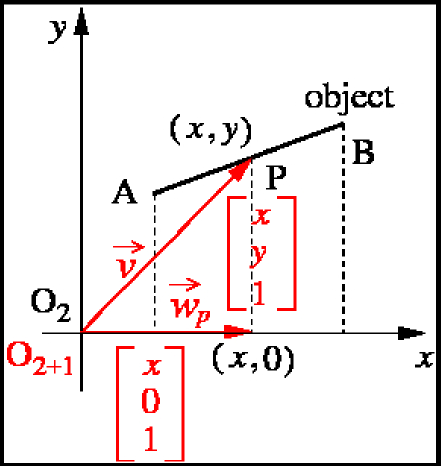
$$\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_p} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projecting a 2D object on the x-axis: a matrix operation



$$\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_p} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\vec{v}} \left(= \begin{bmatrix} \vec{v} \cdot \hat{x} \\ 0 \\ 1 \end{bmatrix} \right)$$

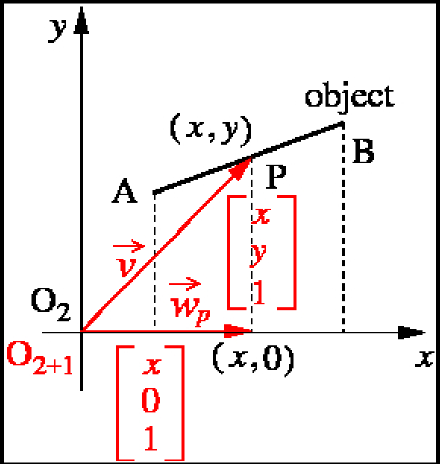
Projecting a 2D object on the x-axis: a matrix operation



$$\underbrace{\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_p} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_p} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\vec{v}} \left(= \begin{bmatrix} \vec{v} \cdot \hat{x} \\ 0 \\ 1 \end{bmatrix} \right)$$

Note: $\vec{w}_p = \vec{v} - (\vec{v} \cdot \hat{y})\hat{y}$

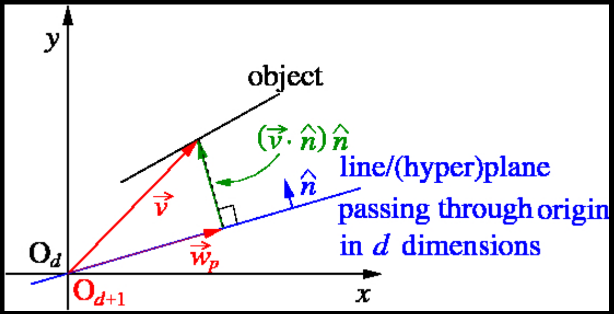
Projecting a 2D object on the x-axis: a matrix operation



$$\underbrace{\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_p} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_p} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\vec{v}} \left(= \begin{bmatrix} \vec{v} \cdot \hat{x} \\ 0 \\ 1 \end{bmatrix} \right)$$

Note: $\vec{w}_p = \vec{v} - (\vec{v} \cdot \hat{y})\, \hat{y}$; i.e., $\vec{w}_p = \begin{bmatrix} \vec{w}_p \cdot \hat{x} \\ \vec{w}_p \cdot \hat{y} \\ \vec{w}_p \cdot \hat{f} \end{bmatrix} = \begin{bmatrix} \vec{v} \cdot \hat{x} \\ 0 \\ 1 \end{bmatrix}$

Projecting an object: a matrix operation

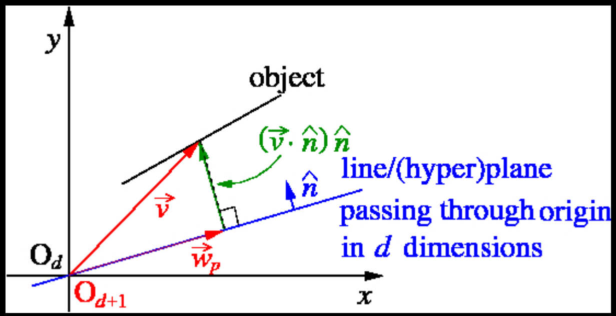


$$\vec{w}_p = \vec{v} - (\vec{v}\cdot\hat{n})\,\hat{n};\ \text{for } d=3,\ \vec{w}_p = \begin{bmatrix} \vec{w}_p\cdot\hat{x} \\ \vec{w}_p\cdot\hat{y} \\ \vec{w}_p\cdot\hat{z} \\ \vec{w}_p\cdot\hat{f} \end{bmatrix} = \begin{bmatrix} \vec{v}\cdot\hat{x} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ \vec{v}\cdot\hat{f} \end{bmatrix}$$

(remember: $\hat{n}\cdot\hat{f} = 0$; i.e., $\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$); $n_x = \hat{n}\cdot\hat{x}$ etc.

Q. What is M_p?

Projecting an object: a matrix operation

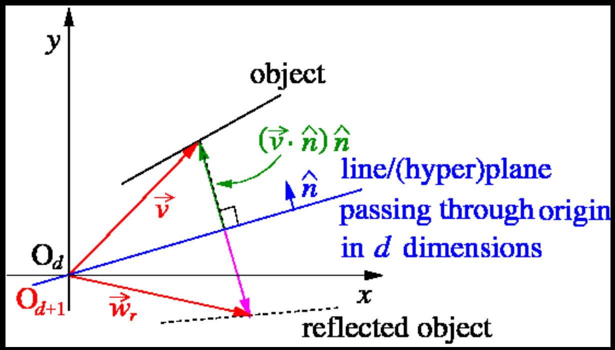


$$\vec{w}_p = \vec{v} - (\vec{v}\cdot\hat{n})\,\hat{n};\ \text{for } d = 3,\ \vec{w}_p = \begin{bmatrix} \vec{w}_p\cdot\hat{x} \\ \vec{w}_p\cdot\hat{y} \\ \vec{w}_p\cdot\hat{z} \\ \vec{w}_p\cdot\hat{f} \end{bmatrix} = \begin{bmatrix} \vec{v}\cdot\hat{x} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ \vec{v}\cdot\hat{f} \end{bmatrix}$$

$$\text{A. } M_p = \begin{bmatrix} 1-n_x^2 & -n_xn_y & -n_xn_z & 0 \\ -n_xn_y & 1-n_y^2 & -n_yn_z & 0 \\ -n_xn_z & -n_yn_z & 1-n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

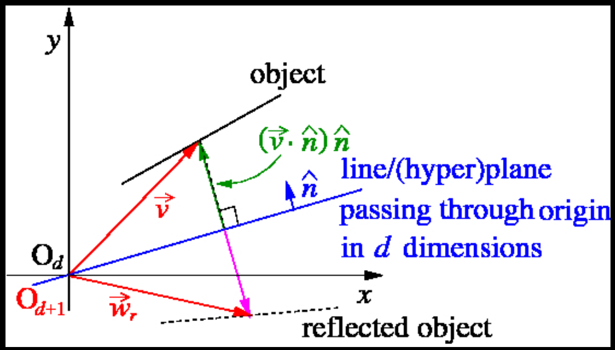
Reflection

Reflecting an object: a matrix operation



Q. $\vec{w}_r$?

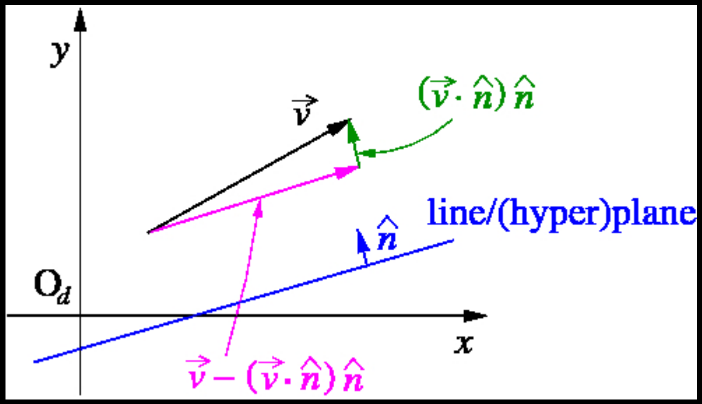
Reflecting an object: a matrix operation



A. $\vec{w}_r = \vec{v} - 2(\vec{v}\cdot\hat{n})\hat{n}$; for $d = 3$, $\vec{w}_r = \begin{bmatrix} \vec{v}\cdot\hat{x} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ \vec{v}\cdot\hat{f} \end{bmatrix}$

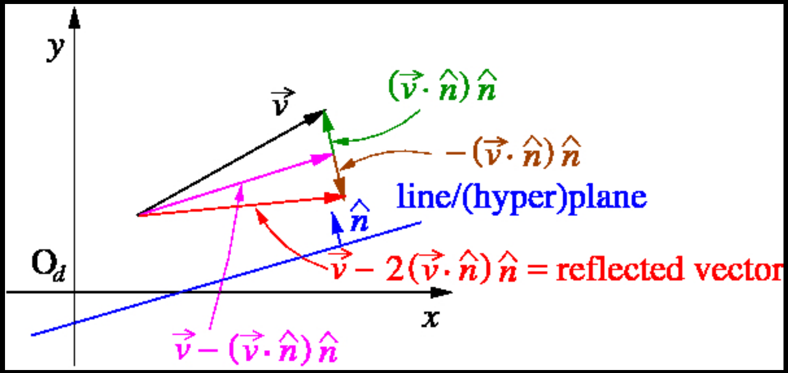
$$\vec{w}_r = M_r\vec{v};\ M_r = \begin{bmatrix} 1-2n_x^2 & -2n_xn_y & -2n_xn_z & 0 \\ -2n_xn_y & 1-2n_y^2 & -2n_yn_z & 0 \\ -2n_xn_z & -2n_yn_z & 1-2n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflecting a vector: a matrix operation



Q. $\vec{v}_r$?

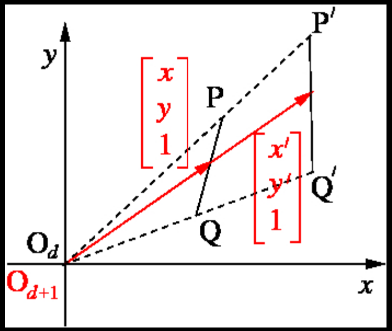
Reflecting a vector: a matrix operation



A. $\vec{v}_r = \vec{v} - 2(\vec{v}\cdot\hat{n})\hat{n}$; e.g. $d = 3$, $\vec{v}_r = \begin{bmatrix} \vec{v}\cdot\hat{x} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ 0 \end{bmatrix} = M_r\vec{v}$

Scaling

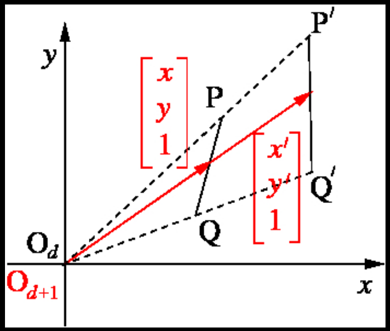
Scaling: a matrix operation



- $d = 2$: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{sc}(\vec{s})} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$; uniform scaling: $s_x = s_y$

  $\vec{s} = \begin{bmatrix} s_x \\ s_y \end{bmatrix}$

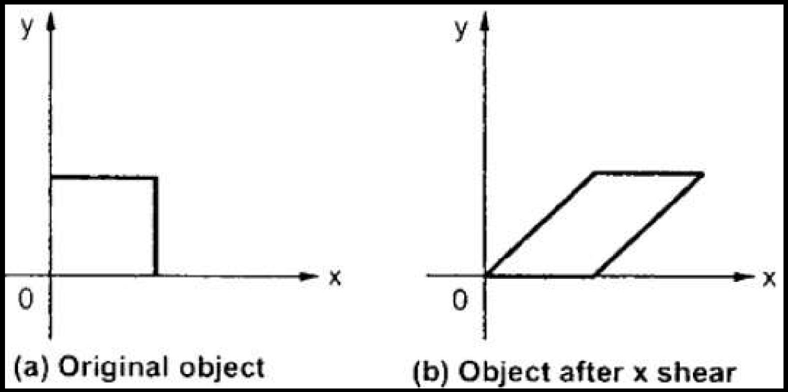
Scaling: a matrix operation

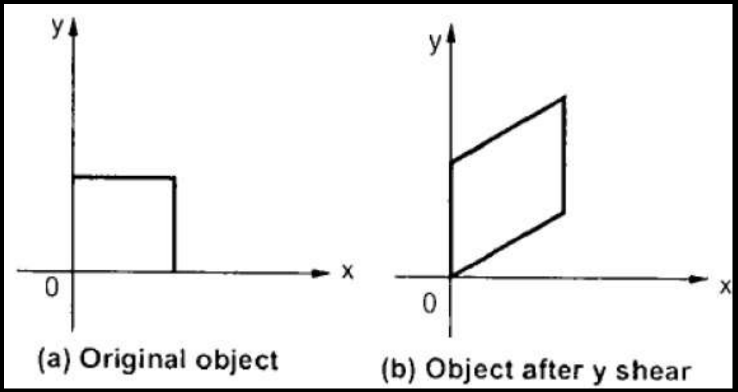


- $d=2$: $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{sc}(\vec{s})} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$; uniform scaling: $s_x = s_y$
- $d=3$: $\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$; uniform scaling: $s_x = s_y = s_z$
- ...

Shearing

Shearing: a matrix operation



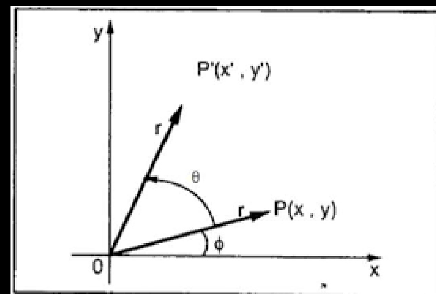


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{M_{sh}(b)} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

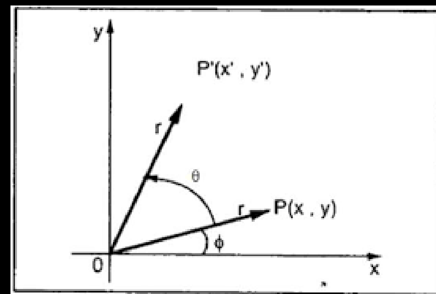
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation

Rotation: a matrix operation [with $M_{ro}(\theta)$]



Rotation: a matrix operation [with $M_{ro}(\theta)$]



3D: Rotation matrix

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$ **X Rotation**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$ **Y Rotation**

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$ **Z Rotation**

Linear (point) transformations

Linear (point) transformations

- Point $(x_1, x_2, \ldots, x_d)$ represented as a vector $\vec{v}$ drawn from the origin
- Linear point transformations satisfy the two following criteria:
  (a) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
  (b) $T(\alpha\vec{v}) = \alpha T(\vec{v})$
- Examples: translation, projection, reflection, ...

Linear (point) transformations

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  (a) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
  (b) $T(\alpha\vec{v}) = \alpha T(\vec{v})$
- Examples: translation, projection, reflection, ...
- Now you can see why this constitutes linear algebra:
  - points/lines/(hyper)planes $\rightarrow$ points/lines/(hyper)planes
    (also known as affine transformation)

Combining transformations

Rule for combining transformations

- Two transformations: e.g., first scale and then rotate

Rule for combining transformations

- Two transformations: e.g., $\underbrace{\text{first scale}}_{\substack{\text{transformation}\\ \text{matrix } M_{sc}}}$ and $\underbrace{\text{then rotate}}_{\substack{\text{transformation}\\ \text{matrix } M_{ro}}}$

  $(x, y, z) \rightarrow (x'', y'', z'')$

Q. How do we express this as a matrix operation?

Rule for combining transformations

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  $(x, y, z) \to (x'', y'', z'')$

Q. How do we express this as a matrix operation?

Hint: $(x, y, z) \to (x', y', z') \to (x'', y'', z'')$

Rule for combining transformations

- Two transformations: e.g., $\underbrace{\text{first scale}}_{\substack{\text{transformation} \\ \text{matrix } M_{sc}}}$ and $\underbrace{\text{then rotate}}_{\substack{\text{transformation} \\ \text{matrix } M_{ro}}}$

  $(x, y, z) \rightarrow (x'', y'', z'')$

Q. How do we express this as a matrix operation?

A. $(x, y, z) \underbrace{\rightarrow}_{M_{sc}} (x', y', z') \underbrace{\rightarrow}_{M_{ro}} (x'', y'', z'')$

Rule for combining transformations

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Q$'$. $(x, y, z) \underbrace{\rightarrow}_{M} (x'', y'', z'')$. Is $M = M_{sc}M_{ro}$ or $M = M_{ro}M_{sc}$?

Rule for combining transformations

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A. $(x, y, z) \underbrace{\rightarrow}_{M_{sc}} (x', y', z') \underbrace{\rightarrow}_{M_{ro}} (x'', y'', z'')$

Q′. $(x, y, z) \underbrace{\rightarrow}_{M} (x'', y'', z'')$. Is $M = M_{sc}M_{ro}$ or $M = M_{ro}M_{sc}$?

A′. $M = M_{ro}M_{sc}$ (order important!)

Remember: for matrices AB is not necessarily $= BA$!

(e.g., y-rotation after x-rotation $\not\equiv$ x-rotation after y-rotation)

Transforming back

Transforming back

- So far, $(x, y, z) \underbrace{\rightarrow}_{M} (x', y', z')$

  now we want $(x', y', z') \underbrace{\rightarrow}_{M'} (x, y, z)$

Q. Given M, how do we calculate M'?

Transforming back

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A. $M' = M^{-1}$

Transforming back

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A. $M' = M^{-1}$

- Sometimes the inverse may not exist (e.g., for projection)

  in that case, M is?

Transforming back

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- Sometimes the inverse may not exist (e.g., for projection)

  in that case, M is a singular matrix

Transforming back, and rotation matrices

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- Sometimes the inverse may not exist (e.g., for projection)

  in that case, M is a singular matrix

- Inverse calculation is expensive (cofactors, determinants...)

  it is however cheap for rotation matrices, since $M_{ro}M_{ro}^{\mathrm{T}} = M_{ro}^{\mathrm{T}}M_{ro}$,

  i.e., simply $M_{ro}^{-1} = M_{ro}^{\mathrm{T}}$

Transforming back, and rotation matrices

- So far, $(x, y, z) \underbrace{\rightarrow}_{M} (x', y', z')$

  now we want $(x', y', z') \underbrace{\rightarrow}_{M'} (x, y, z)$

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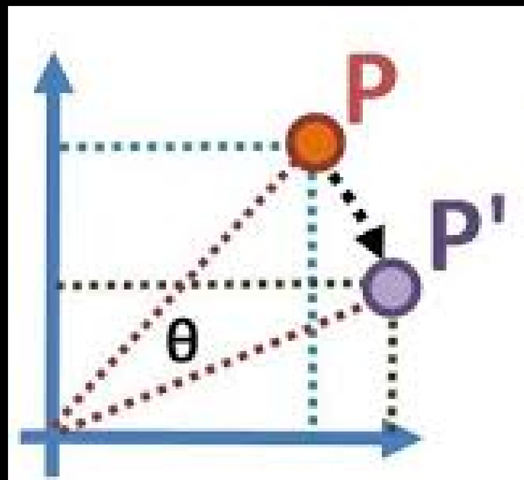
  it is however cheap for rotation matrices, since $M_{ro}M_{ro}^{\mathrm{T}} = M_{ro}^{\mathrm{T}}M_{ro}$,

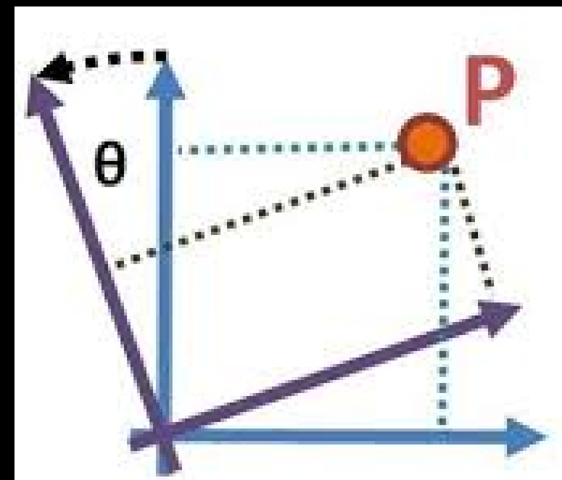
  i.e., simply $M_{ro}^{-1} = M_{ro}^{\mathrm{T}}$; note: $M_{ro}^{-1}(\theta) = M_{ro}^{\mathrm{T}}(\theta) = M_{ro}(-\theta)$

Co-ordinate, or passive transformations

Active vs. passive transformations (with rotation)



point transformation
(aka active transformation)



co-ordinate transformation
(aka passive transformation)

- Co-ordinate 2 is anti-clockwise rotated (θ) wrt co-ordinate 1 means $(x_1, y_1, z_1) \underbrace{\rightarrow}_{M_{ro}(-\theta)} (x_2, y_2, z_2)$ describes the same physical point

Active vs. passive transformations: translation

- Co-ordinate 2 is translated ($\vec{a}$) wrt co-ordinate 1
  means $(x_1, y_1, z_1) \underbrace{\rightarrow}_{M_t(-\vec{a})} (x_2, y_2, z_2)$ describes the same physical point

Active vs. passive transformations: translation and scaling

- Co-ordinate 2 is translated ($\vec{a}$) wrt co-ordinate 1

  means $(x_1, y_1, z_1) \underbrace{\rightarrow}_{M_t(-\vec{a})} (x_2, y_2, z_2)$ describes the same physical point

- Co-ordinate 2 is scaled ($\vec{s}$) wrt co-ordinate 1

  means $(x_1, y_1, z_1) \underbrace{\rightarrow}_{M_{sc}(\overrightarrow{1/s})} (x_2, y_2, z_2)$ describes the same physical point

Summary

- Point, or active transformations
  - translation
  - projection
  - reflection
  - scaling
  - shearing
  - rotation
  - linear transformation properties
  - combining transformations (and transformation back!)
- Co-ordinate, or passive transformations
- Next class: viewing transformations

Finally, references...

- Book chapter 6: Transformation matrices (can leave out Sec. 6.5)