Graphics (INFOGR), 2018-19, Block IV, lecture 12

Deb Panja

Today: Matrix reloaded 2

(or Viewing Transformations)

Welcome

Note for today's lecture

- Will be closely following book chapter 7
  (read it thoroughly to grab the concepts)

- Most of the lecture will be in symbols

Today

- Redo part of last lecture
- Rotation co-ordinate transformation revisited
- Viewing transformation (getting world space $\rightarrow$ screen space)
  - viewport transfomation
  - orthographic tranformation
  - camera transformation
  - projection transformation
  - "graphics pipeline": putting everything together
- Summary maths lectures

Redo part of last lecture

From last lecture

- Point, or active transformations
  - translation
  - projection redo
  - reflection
  - scaling
  - shearing
  - rotation
  - linear transformation properties
  - combining transformations (and transformation back!)
- Co-ordinate, or passive transformations

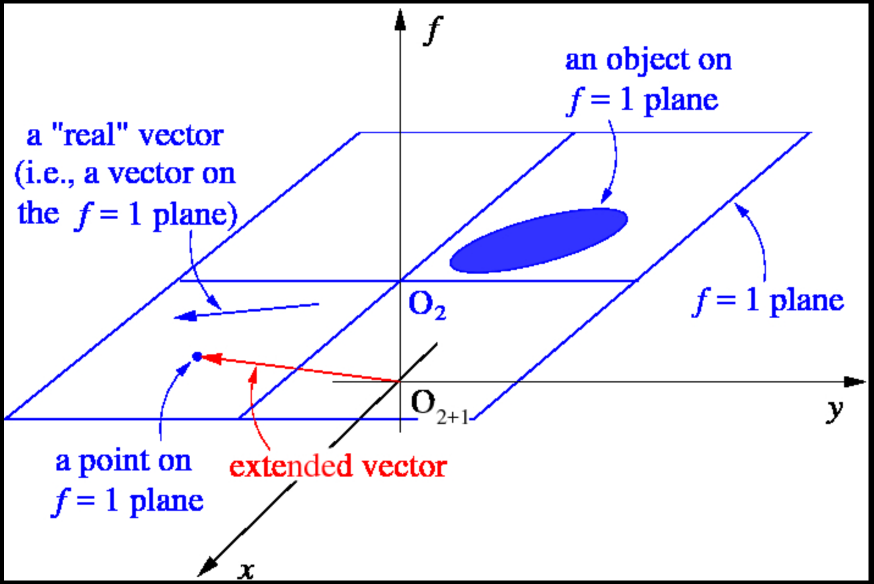
Translation: a matrix operation

- We translate a point P (x, y, z) by (a_x, a_y, a_z)

  i.e., $x' = x + a_x$, $y' = y + a_y$, $z' = z + a_z$

$$\underbrace{\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}}_{\vec{w}_t} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_x \\ 0 & 1 & 0 & a_y \\ 0 & 0 & 1 & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_t(\vec{a})} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\vec{v}}; \; \vec{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

How to think about an extended vector for 2D

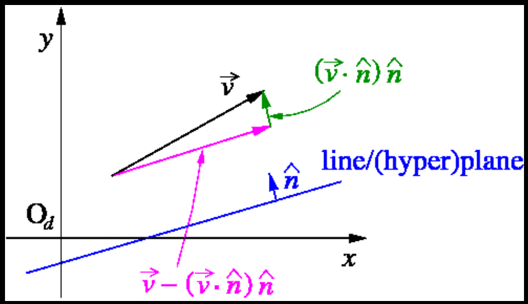


- Note: A "real" vector $\vec{v}$, by construction, satisfies $\vec{v} \cdot \hat{f} = 0$

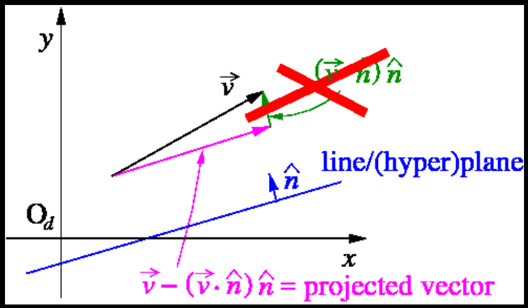
  e.g., the (2+1)D representation of a real vector in 2D is $\begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$

Projecting and reflecting vectors

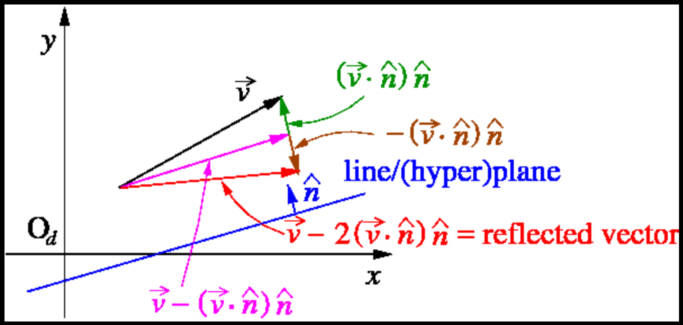
Projecting and reflecting vectors



original vector



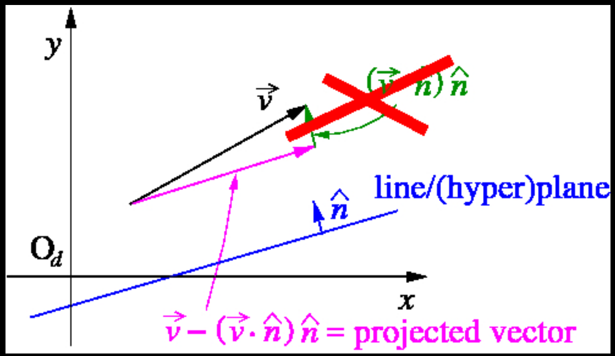
projected vector



reflected vector

These rules always hold!

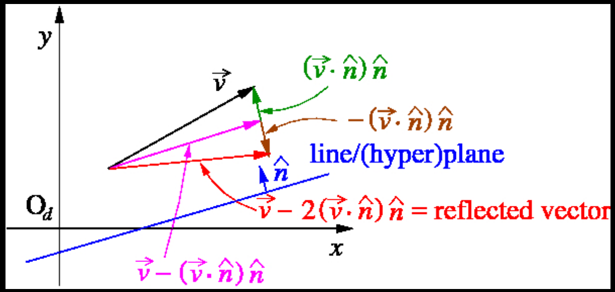
Projecting a vector: a matrix operation in (3+1)D



$$\vec{w}_p = \vec{v} - (\vec{v}\cdot\hat{n})\,\hat{n};\ \text{for } d=3,\ \vec{w}_p = \begin{bmatrix} \vec{w}_p\cdot\hat{x} \\ \vec{w}_p\cdot\hat{y} \\ \vec{w}_p\cdot\hat{z} \\ \vec{w}_p\cdot\hat{f} \end{bmatrix} = \begin{bmatrix} \vec{v}\cdot\hat{x} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - (\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ \vec{v}\cdot\hat{f} \end{bmatrix}$$

$$\vec{w}_p = M_p\,\vec{v};\ M_p = \begin{bmatrix} 1-n_x^2 & -n_xn_y & -n_xn_z & 0 \\ -n_xn_y & 1-n_y^2 & -n_yn_z & 0 \\ -n_xn_z & -n_yn_z & 1-n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflecting a vector: a matrix operation in (3+1)D



$$\vec{w}_r = \vec{v} - 2(\vec{v}\cdot\hat{n})\hat{n};\ \text{for } d = 3,\ \vec{w}_r = \begin{bmatrix} \vec{v}\cdot\hat{x} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{x}) \\ \vec{v}\cdot\hat{y} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{y}) \\ \vec{v}\cdot\hat{z} - 2(\vec{v}\cdot\hat{n})(\hat{n}\cdot\hat{z}) \\ \vec{v}\cdot\hat{f} \end{bmatrix}$$

$$\vec{w}_r = M_r\,\vec{v};\ M_r = \begin{bmatrix} 1-2n_x^2 & -2n_xn_y & -2n_xn_z & 0 \\ -2n_xn_y & 1-2n_y^2 & -2n_yn_z & 0 \\ -2n_xn_z & -2n_yn_z & 1-2n_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projecting and reflecting points

Projecting and reflecting points (on an object)

- Can we use vector projection/reflection formulae for points as well?

  yes, provided care is taken

Projecting and reflecting points (on an object)

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  yes, provided care is taken

- Why?

  because specifying a vector (arrow) does not specify its starting point;

  although point P (x, y, z) is reached by vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ from the origin,

  the origin may "move" upon projection/reflection

Projecting points (on an object)

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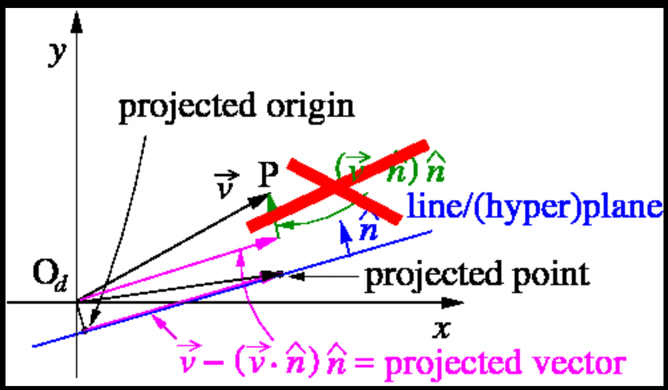
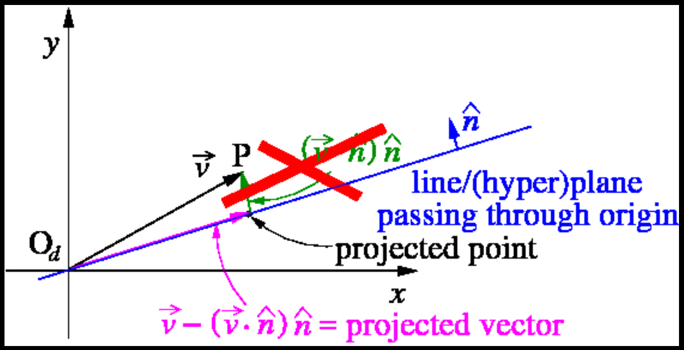
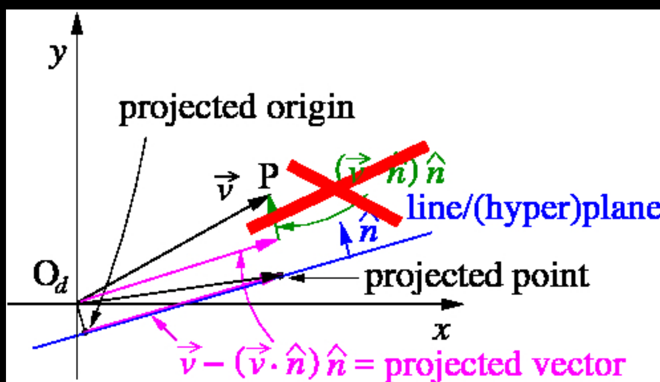


$\longleftrightarrow$ contrast



Projecting and reflecting points (on an object)

- Specifying a vector (arrow) does not specify its starting point; although point P (x, y, z) is reached by vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ from the origin, the origin may "move" upon projection/reflection



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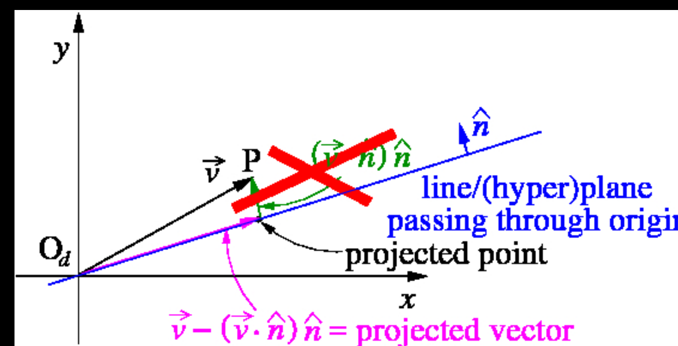
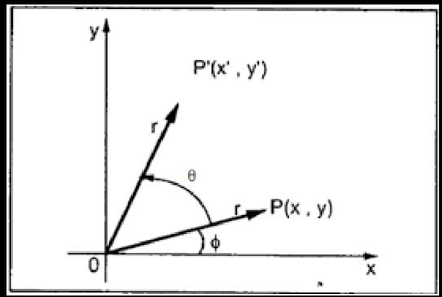


- The case for reflection is similar (not shown further)

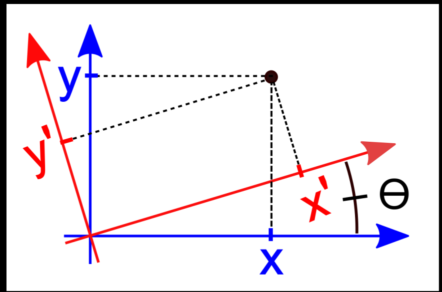
Rotation co-ordinate transformation revisited

Active rotation in (2+1)D revisited

- Active: $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



- Now consider the passive (co-ordinate) rotation



Active rotation in (2+1)D revisited

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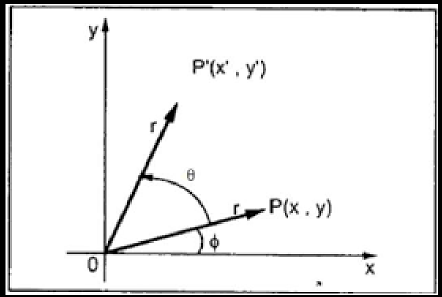

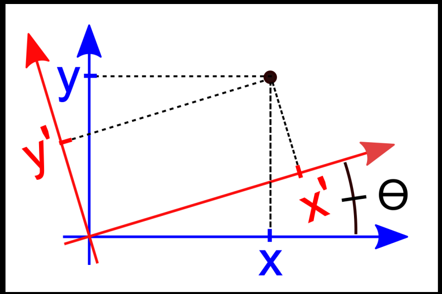

- Now consider the passive (co-ordinate) rotation

Q. $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = M_{ro} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$; $M_{ro} = ?$

Active rotation in (2+1)D revisited

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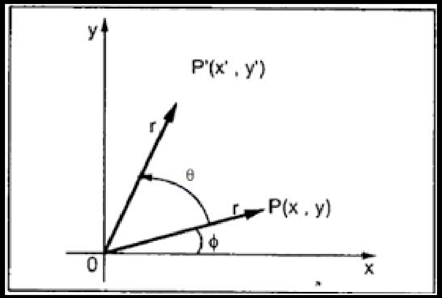

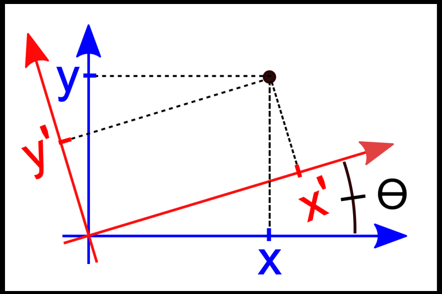

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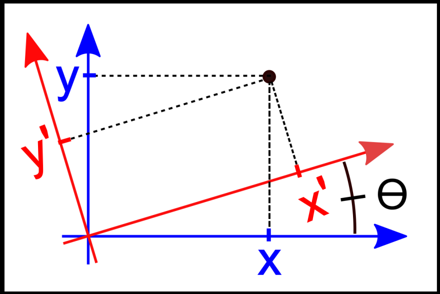
A. $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Passive rotation in (2+1)D

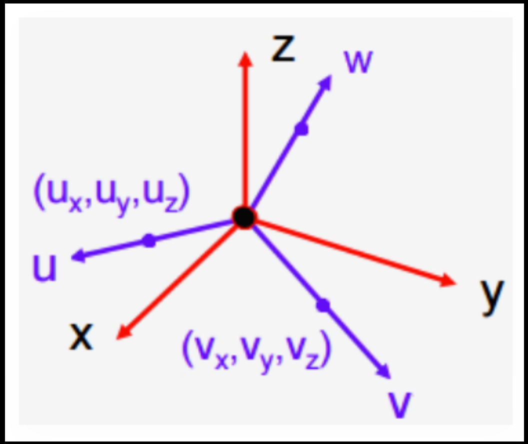
- $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



$\cos\theta = \hat{x}\cdot\hat{x}' = \hat{y}\cdot\hat{y}'$, $\sin\theta = \hat{y}\cdot\hat{x}' = -\hat{x}\cdot\hat{y}'$

then $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{x}'\cdot\hat{x} & \hat{x}'\cdot\hat{y} & 0 \\ \hat{y}'\cdot\hat{x} & \hat{y}'\cdot\hat{y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_{x'} & y_{x'} & 0 \\ x_{y'} & y_{y'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

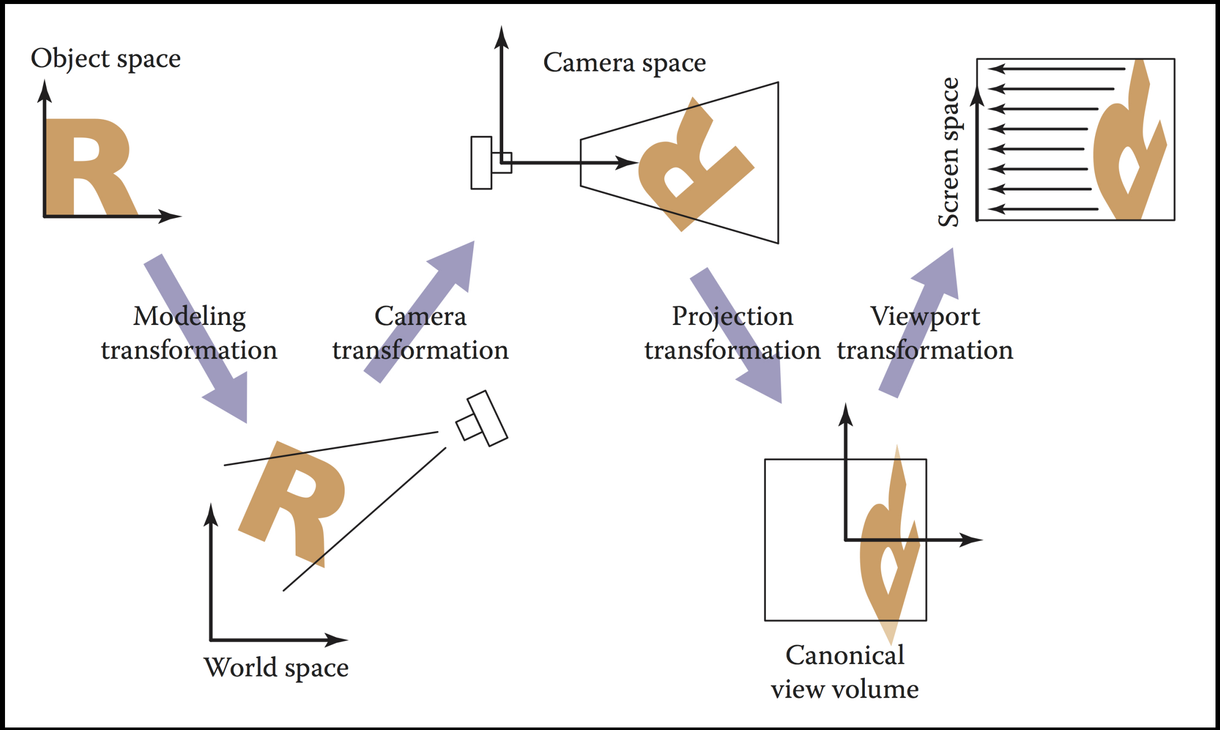
Passive rotation in (3+1)D



- $$\begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{u}\cdot\hat{x} & \hat{u}\cdot\hat{y} & \hat{u}\cdot\hat{z} & 0 \\ \hat{v}\cdot\hat{x} & \hat{v}\cdot\hat{y} & \hat{v}\cdot\hat{z} & 0 \\ \hat{w}\cdot\hat{x} & \hat{w}\cdot\hat{y} & \hat{w}\cdot\hat{z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_w & y_w & z_w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Viewing transformation

What is viewing transformation?



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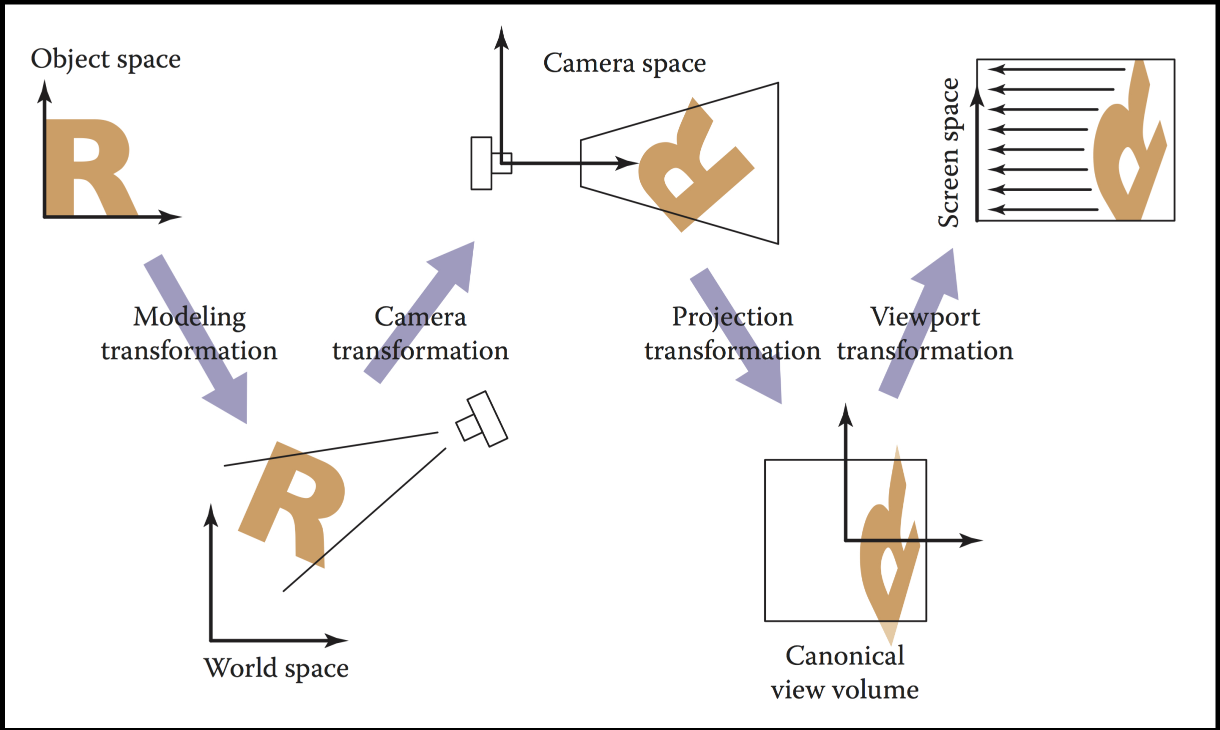
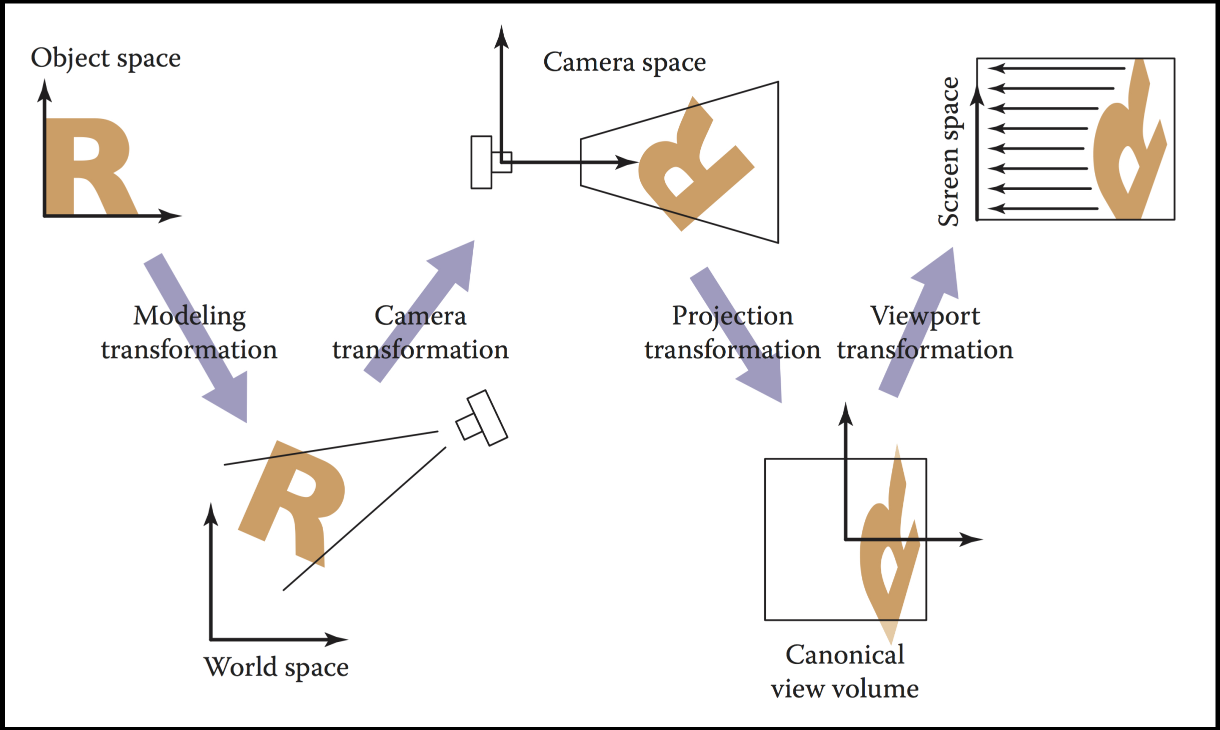


- Will achieve these (passive!) transformations by concatenating matrices

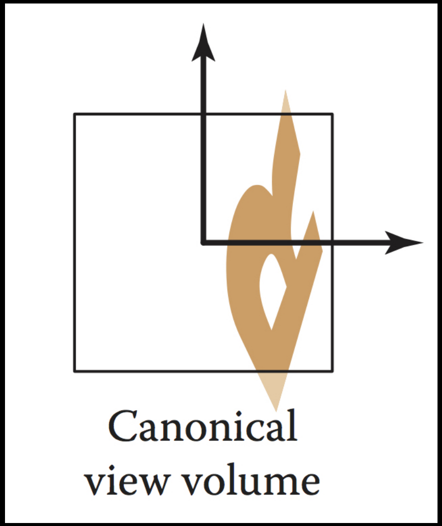
What is viewing transformation?



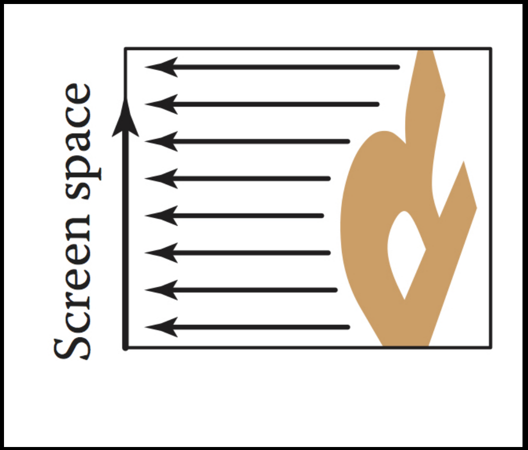
- Will achieve these (passive!) transformations by concatenating matrices (and we need to do that in the reverse order of transformations)

Viewport transformation

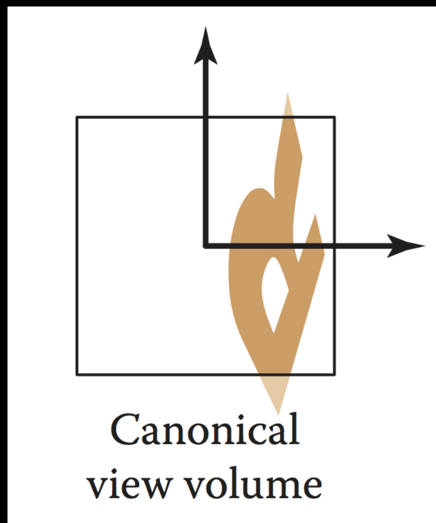
Viewport transformation



$$\xrightarrow[M_{\mathrm{vp}}]{}$$



Viewport transformation



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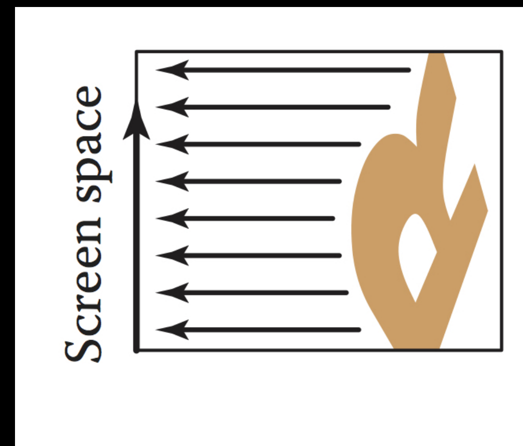

- Canonical view space: $(x, y, z) \in [-1, 1]^3$
- Screen space: $n_x \times n_y$ pixels
- M_{vp}: transform $[-1, 1]^2 \to [-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$
  (only for x and y, don't care about z)

Viewport transformation: M_{vp}

- M_{vp}: transform $[-1,1]^2 \to [-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$
  (only for x and y, don't care about z)

- Concatenate translation after scaling (diagonal matrix):

$$
\begin{array}{cc}
\begin{array}{cccc} & & \downarrow & \phantom{\frac{n_x-1}{2}} \end{array} & \begin{array}{cccc} \phantom{\frac{n_x}{2}} & \downarrow & & \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & \frac{n_x-1}{2} \\ 0 & 1 & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & 0 \\ 0 & \frac{n_y}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

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$$\overset{\downarrow}{\begin{bmatrix} 1 & 0 & 0 & \frac{n_x-1}{2} \\ 0 & 1 & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \overset{\downarrow}{\begin{bmatrix} \frac{n_x}{2} & 0 & 0 & 0 \\ 0 & \frac{n_y}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

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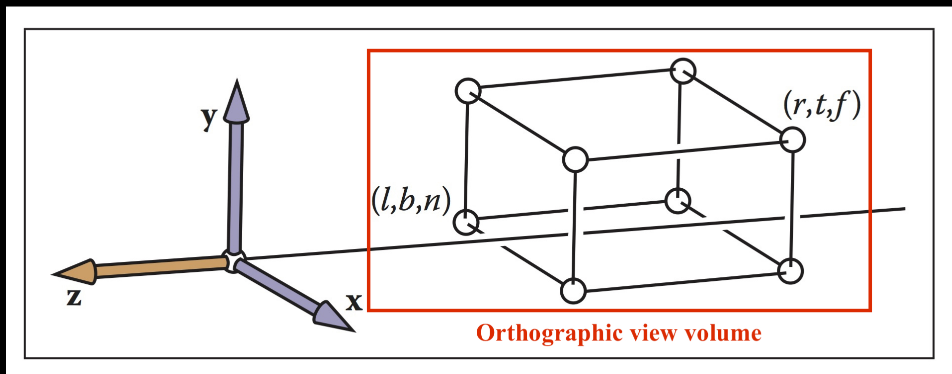
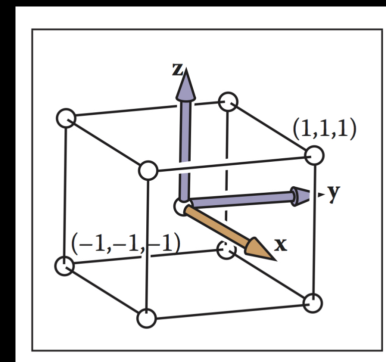
- After concatenation, we obtain $M_{\text{vp}} = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Orthographic projection transformation

Orthographic transformation



$\xrightarrow{\hspace{2em}}$ M_{ortho}



(canonical view volume)

- M_{ortho}: transform $[l, r] \times [b, t] \times [n, f] \to [-1, 1]^3$

Orthographic transformation: M_{ortho}

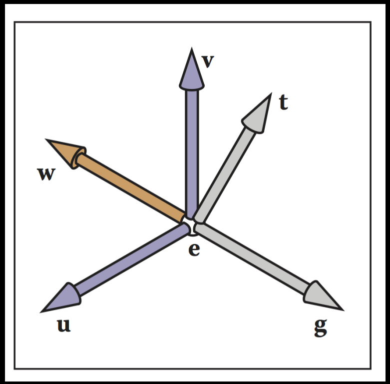
- M_{ortho}: transform $[l, r] \times [b, t] \times [n, f] \to [-1, 1]^3$
- Concatenate scaling (diagonal matrix) after translation:

$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- After concatenation we obtain $M_{\text{ortho}} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

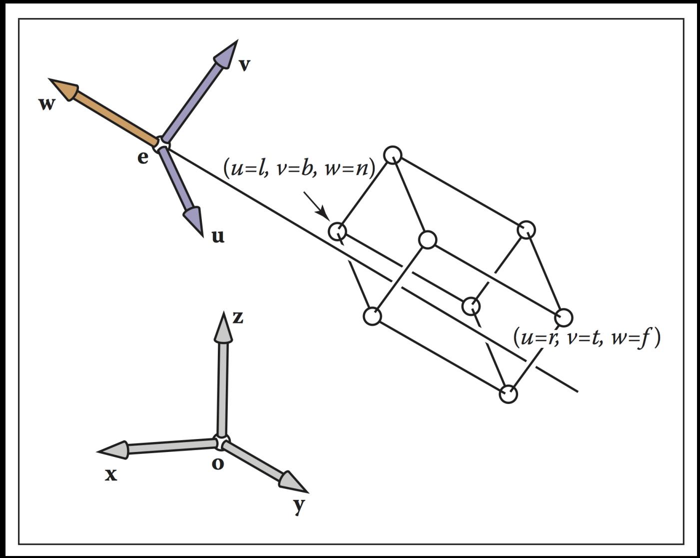
Camera transformation

The camera...



- Camera specifications
  - position $\vec{e}$
  - gaze direction $\hat{g}$; $\hat{w} = -\hat{g}$
  - view up vector $\hat{t}$: any vector that symmetrically bisects the viewer's head into left and right, and points "to the sky"
  - finally, $\hat{u} = (\hat{t} \times \hat{w})/||\hat{t} \times \hat{w}||$, and $\hat{v} = \hat{w} \times \hat{u}$
    $(\hat{u}, \hat{v}, \hat{w})$ forms a right-handed co-ordinate system

World co-ordinates to camera co-ordinates



- M_{cam}: transform $(x, y, z) \rightarrow (u, v, w)$

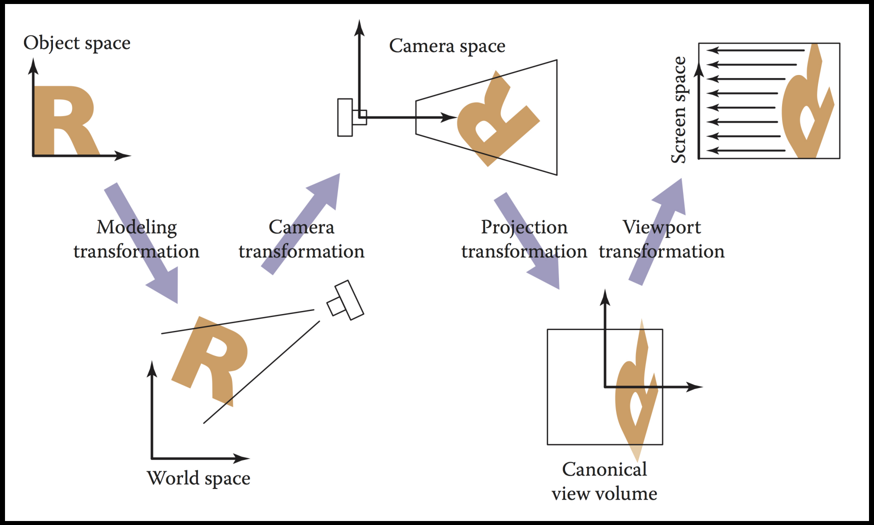
World co-ordinates to camera co-ordinates

- M_{cam}: transform $(x, y, z) \to (u, v, w)$
- Concatenate rotation of axes after translation:

$$\begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_w & y_w & z_w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- After concatenation we obtain M_{cam}

Finally we can put things together



- The "graphics pipeline":

$$\underbrace{\text{world space} \rightarrow \underset{M_{\text{cam}}}{\overset{}{\downarrow}}\ \text{camera space} \rightarrow \underset{M_{\text{ortho}}}{\downarrow}\ \text{canonical view space} \rightarrow \underset{M_{\text{vp}}}{\downarrow}\ \text{screen space}}_{M = M_{\text{vp}}\, M_{\text{ortho}}\, M_{\text{cam}}}$$

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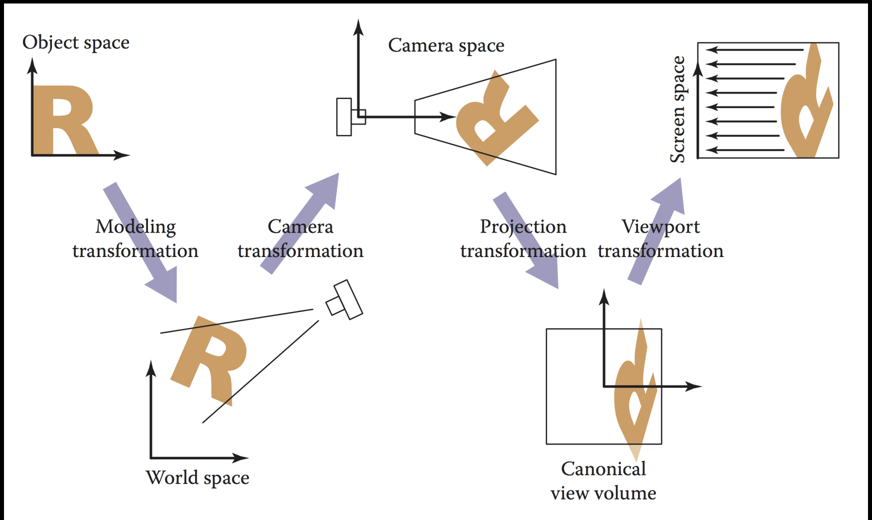
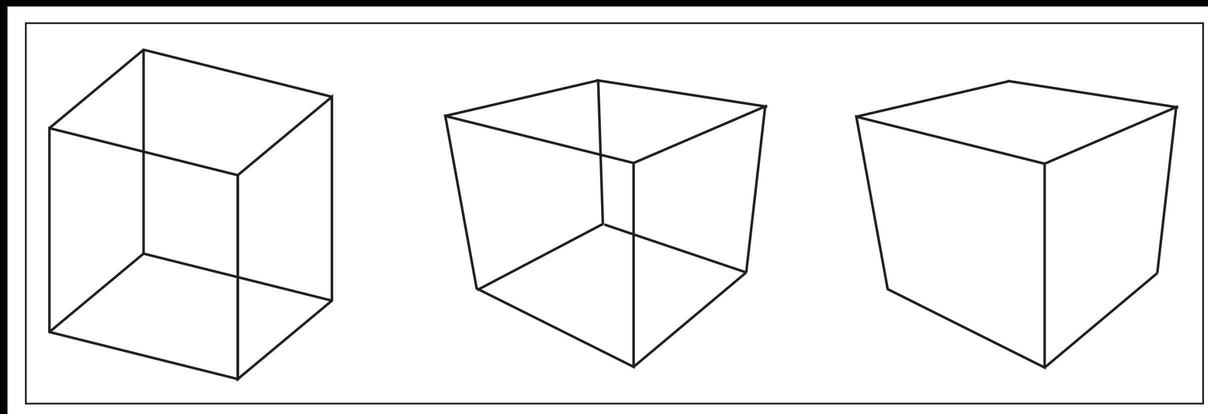


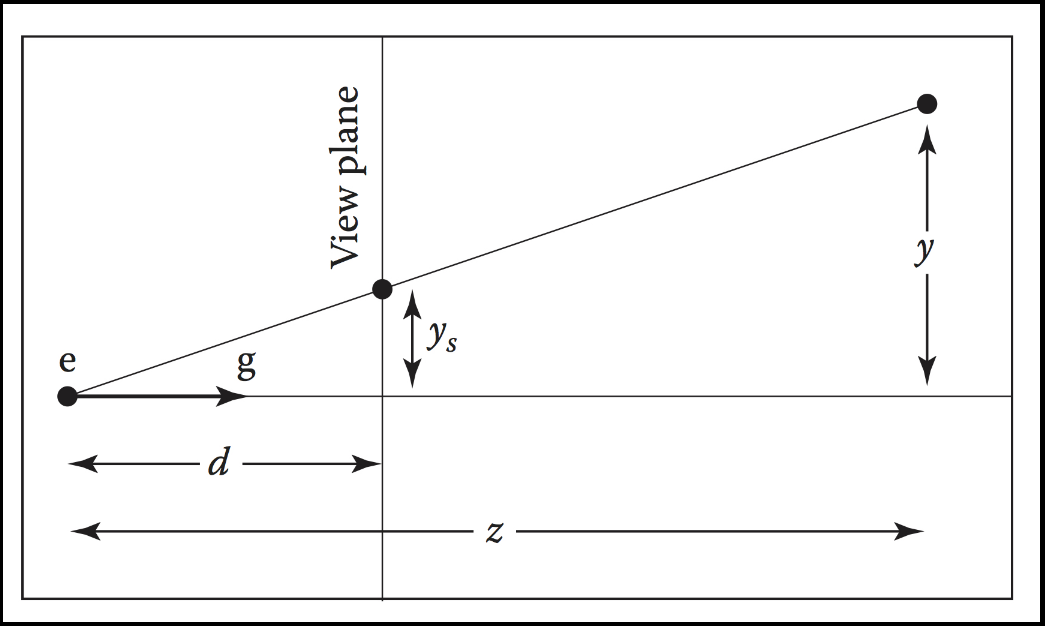
- The "graphics pipeline":

$$\underbrace{\begin{array}{ccccccc} \text{world space} & \rightarrow & \text{camera space} & \rightarrow & \text{canonical view space} & \rightarrow & \text{screen space} \\ & \downarrow & & \downarrow & & \downarrow & \\ & M_{\text{cam}} & & M_{\text{ortho}} & & M_{\text{vp}} & \end{array}}_{M = M_{\text{vp}}\, M_{\text{ortho}}\, M_{\text{cam}};\ \text{no, we are not done yet!}}$$

Because we have missed the perspective effect

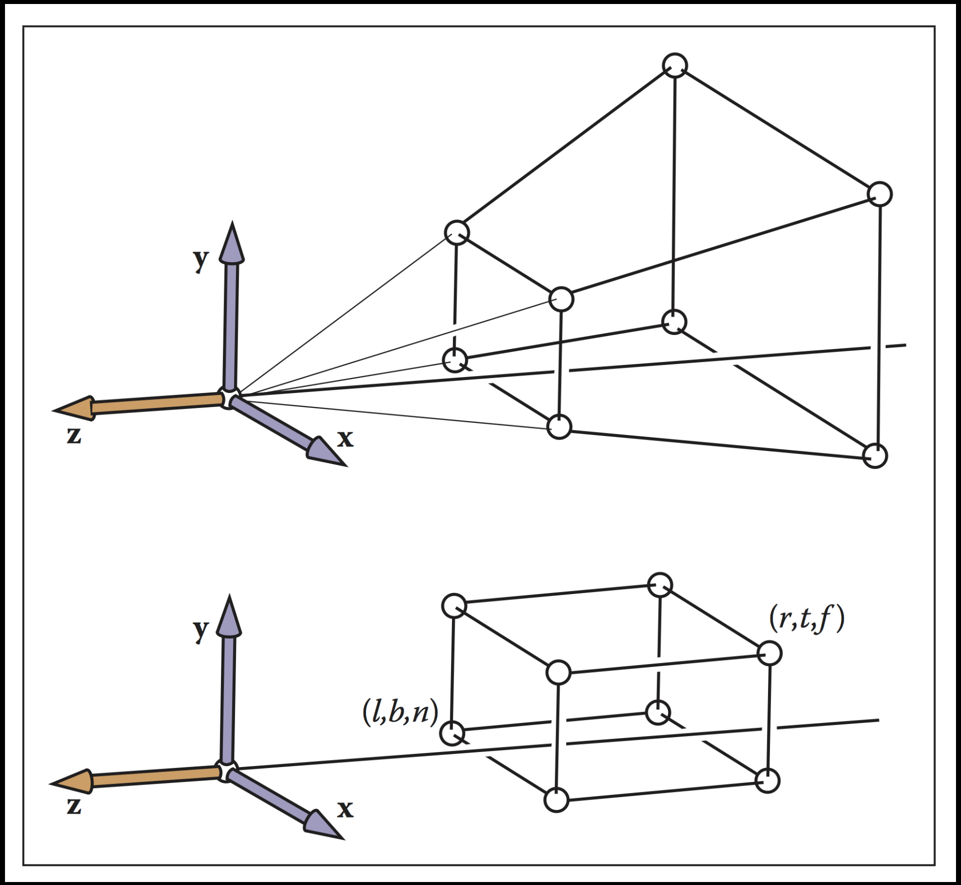


A simpler perspective case

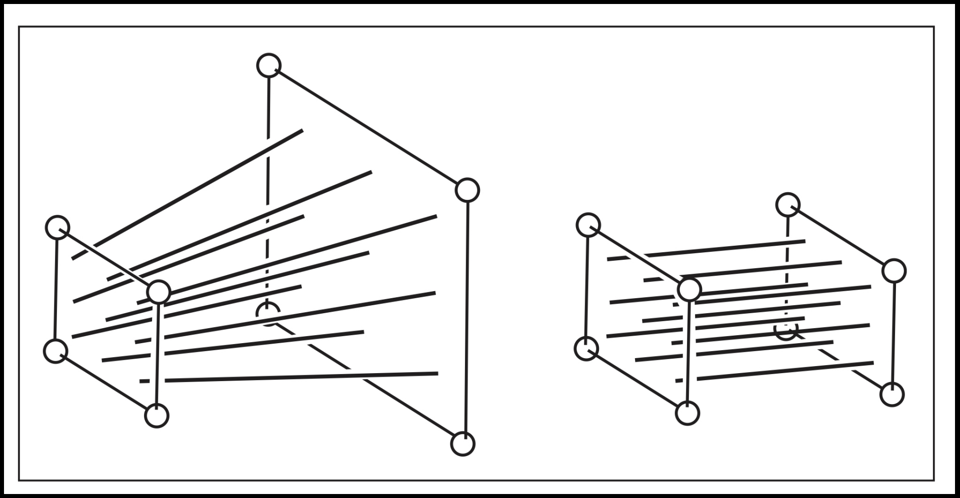


- $y_s = \dfrac{yd}{z}$
  (this is the principle we need to implement in 3D)

Camera field of view

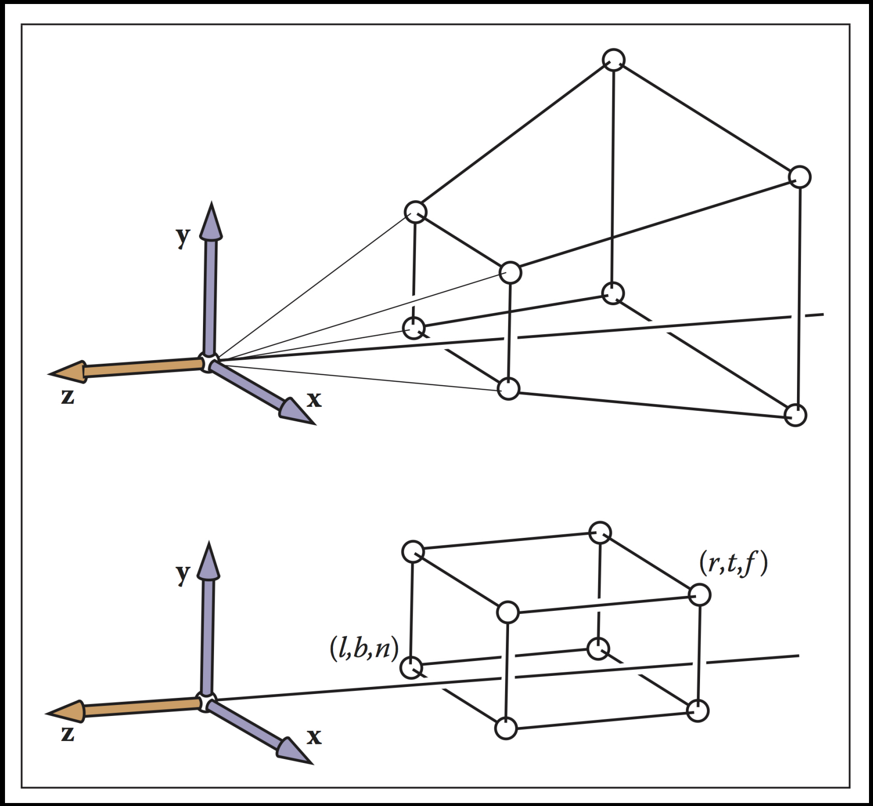


Camera field of view



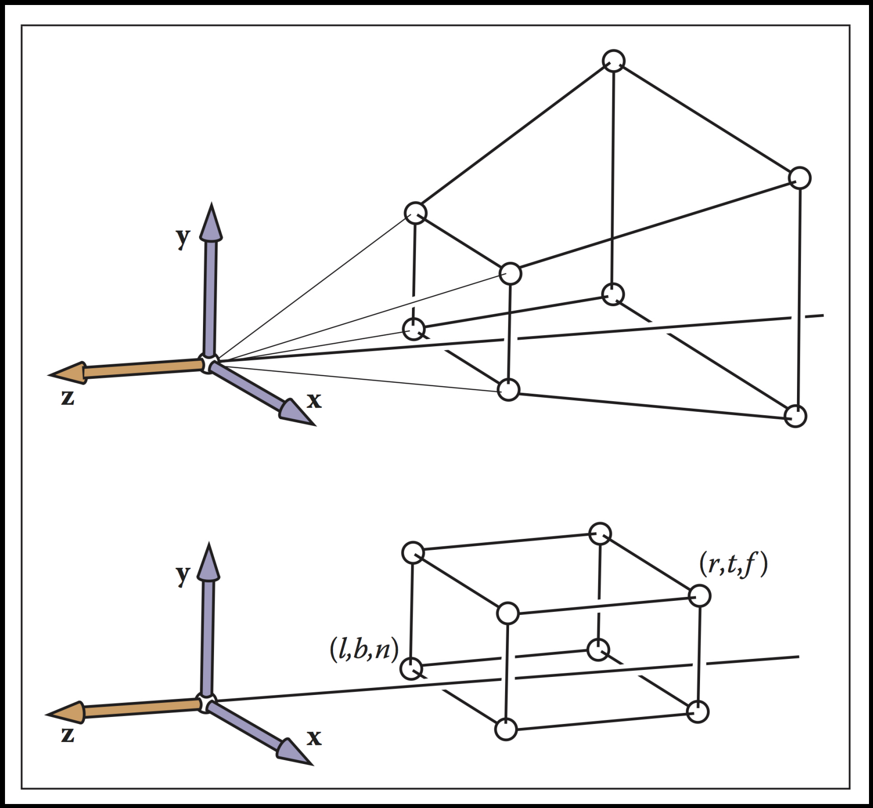
Projection transformation

Projection transformation



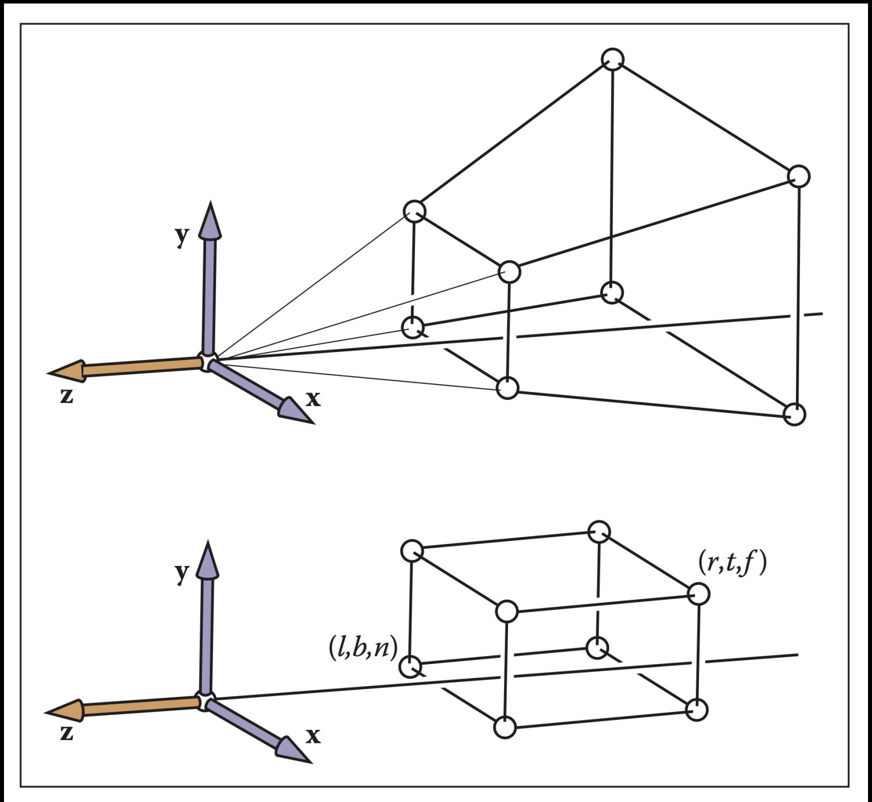
- After projection, we want:
  - projection $P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ ? \\ 1 \end{bmatrix}$

Projection transformation



- After projection, we want:
  - projection $P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ ? \\ 1 \end{bmatrix}$
  - constraints:
    $z = \{n, f\} \xrightarrow[P]{} \{n, f\}$

Projection transformation



- After projection, we want:
  - projection $P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ ? \\ 1 \end{bmatrix}$
  - constraints:
    $z = \{n, f\} \xrightarrow[P]{} \{n, f\}$
- No unique choice for P

Fixing P for projection

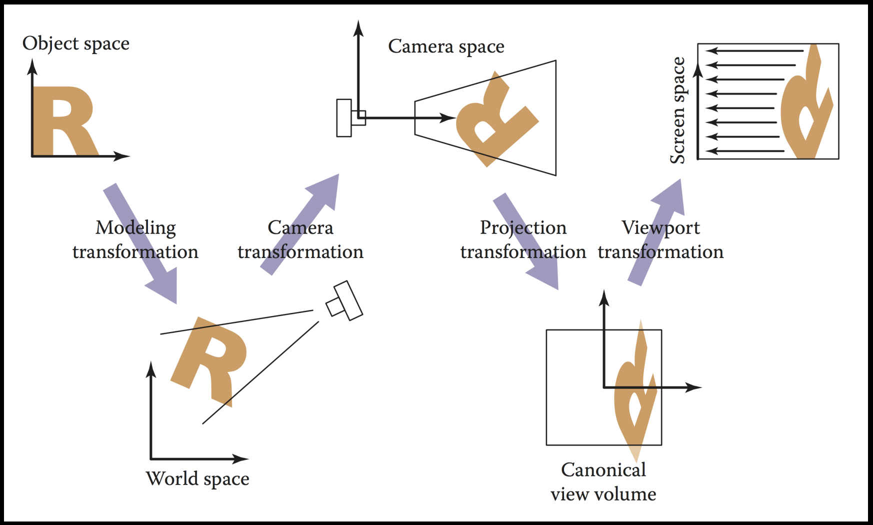
- We want $P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ ? \\ 1 \end{bmatrix}$, with $z = \{n, f\} \underset{P}{\longrightarrow} \{n, f\}$
- Cannot be achieved by a simple matrix multiplication by P

Fixing P for projection: follow the book

- We want $P\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ ? \\ 1 \end{bmatrix}$, with $z = \{n, f\} \underset{P}{\longrightarrow} \{n, f\}$
- Cannot be achieved by a simple matrix multiplication by P
- Choose $P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$;

$$P\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{bmatrix} \sim \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{bmatrix}$$

Finally, the graphics pipeline



- World space $\rightarrow$ camera space $\rightarrow$ canonical view space $\rightarrow$ screen space

$$\underbrace{\overset{\downarrow}{M_{\text{cam}}} \qquad\qquad \overset{\downarrow}{M_{\text{ortho}}} \qquad\qquad\qquad \overset{\downarrow}{M_{\text{vp}}}}_{M = M_{\text{vp}}\, M_{\text{ortho}}\, P\, M_{\text{cam}}}$$

Summary for today's lecture

- Redo part of last lecture
- Rotation co-ordinate transformation revisited
- Viewing transformation (getting world space $\rightarrow$ screen space)
  - viewport transfomation
  - orthographic tranformation
  - camera transformation
  - projection transformation
  - "graphics pipeline": putting everything together

References for today

- Book chapter 6.5: Co-ordinate transformations
- Book chapter 7: Viewing tranformations

Summary maths lectures

Summary maths lecture 1

- Vectors and Vector operations
  - addition and subtraction (requires same dimensionality)
  - scalar multiplication
  - magnitude/length/norm of a vector; unit, null and basis vectors
  - Pythagoras theorem
  - elementary trigonometry: definitions of $\sin, \cos, \tan$

Summary maths lecture 2

- Dot product of vectors
- Shooting rays for a line
  - equations: parametric, slope-intercept and implicit forms
  - parallel and perspective projections of a line
  - projection of a point on a line, tangent and normal vectors

Summary maths lecture 3

- Circles, ellipses and shooting rays at them
- Shooting rays as a line in 3D
  - equations: parametric and implicit-like forms
- Equation of a plane using the normal vector
- Cross product, left- and right-handed co-ordinate systems
  - implicit and parametric equations of a plane
  - tangent and bitangent vectors

Summary maths lecture 4

- projections of a line on a plane
- Spheres and spherical co-ordinate system
  - surface normal and tangent planes
  - shooting rays towards a sphere and their intersections

Summary maths lecture 5

- Why matrices? The operations defined for them make them special
  - matrix dimensions, special matrices (diagonal, identity, null)
- Matrix operations (addition, scalar multiplication, subtraction, matrix multiplication, transpose)
- Determinants (only for square matrices!)
- Adjoint/adjugate and inverse of matrices (only for square matrices!)
- Geometric interpretation of determinants
- Introduction to transformations
  - translation and the fictitious coordinate

Summary maths lecture 6

- Point, or active transformations
  - translation
  - projection
  - reflection
  - scaling
  - shearing
  - rotation
  - linear transformation properties
  - combining transformations (and transformation back!)
- Co-ordinate, or passive transformations

Finally...

- Good luck with the rest of the second half of the course!
- And please do not forget to leave feedback on caracal