Graphics (INFOGR), 2018-19, Block IV, lecture 2

Deb Panja

Today: Vector algebra (continued), primitives and projections in 2D

Welcome

Today

- Dot/scalar product between two vectors
- Lines in 2D
- Projections and shadows in 2D

Recap, and ingredients

- Will work in 2D, Cartesian co-ordinates (x and y reference directions)
- A vector (arrow) has a magnitude (length) and a direction
  - a unit vector has magnitude unity
  - in Cartesian co-ordinates $\hat{x}$ and $\hat{y}$ as basis vectors
- A point and a vector are two fundamentally different entities
  - a point P: (x, y), a vector $\vec{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
  - will use vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to reach point (x, y) from the origin
- Primitives are basically objects, shapes or forms (lines, circles,...)
- Trigonometry: definitions of $\sin, \cos, \tan$

Dot/scalar product between two vectors

- Vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_d \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$

  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = u_1 v_1 + u_2 v_2 + \ldots + u_d v_d$ (a scalar)

Dot/scalar product between two vectors

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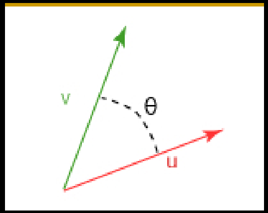
  geometrically, $\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta$

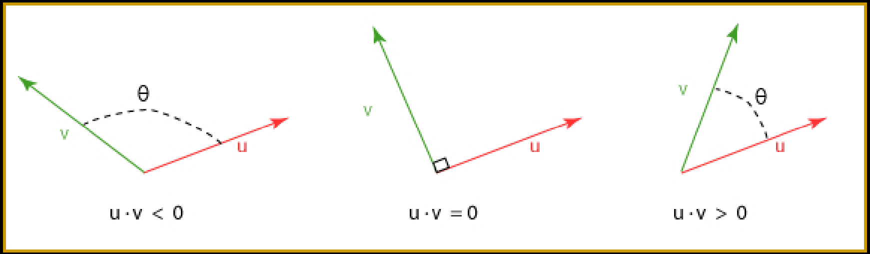


  (notice: $||\vec{u}||^2 = \vec{u} \cdot \vec{u}$; $\cos 0° = 1$)

Dot/scalar product: another geometric interpretation

- $\vec{u} \cdot \vec{v}$: Component of $\vec{v}$ along $\vec{u}$ or vice versa

  ("how much" of $\vec{v}$ is aligned to $\vec{u}$ or vice versa)



Dot/scalar product: another geometric interpretation

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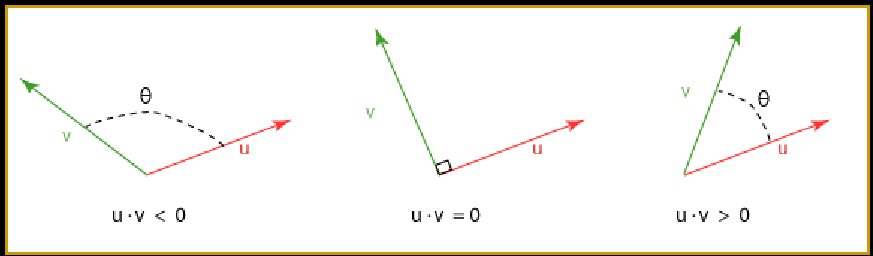
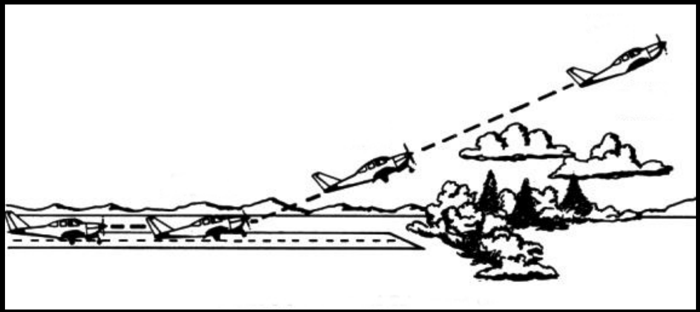


Q. How much of the plane’s actual speed counts for ground speed?



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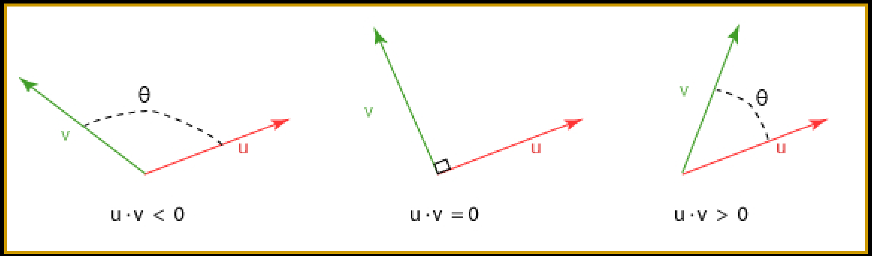
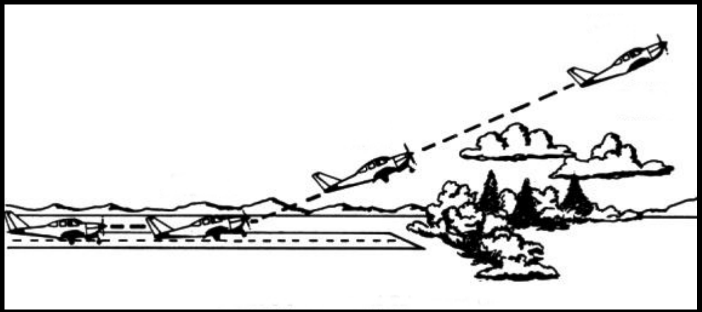
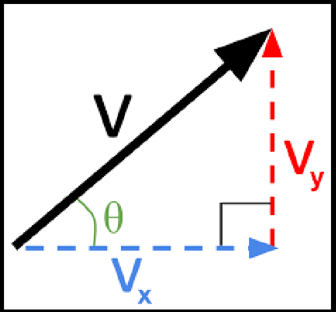




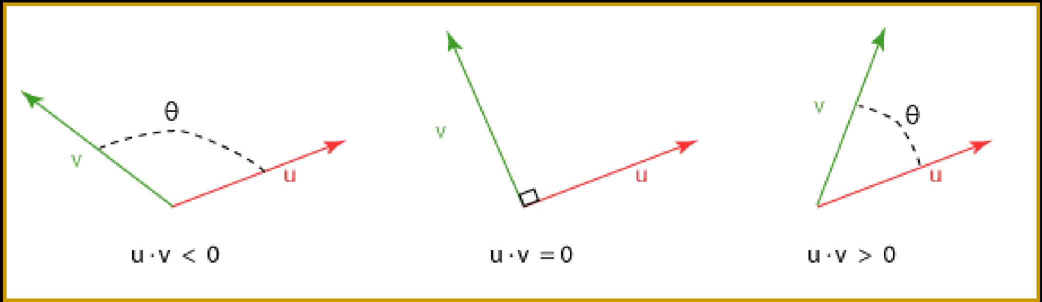


A. Actual speed: $||\vec{v}||$

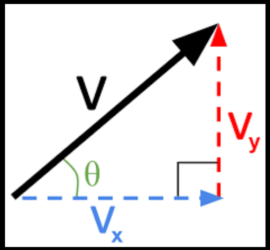
Ground speed:
$||\vec{v}_x|| = ||\vec{v}|| \cos\theta = \vec{v} \cdot \hat{x}$

Dot/scalar product: another geometric interpretation

- $\vec{u} \cdot \vec{v}$: Component of $\vec{v}$ along $\vec{u}$ or vice versa
  ("how much" of $\vec{v}$ is aligned to $\vec{u}$ or vice versa)



- for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_d \end{bmatrix}$, v_i is the i-th component of $\vec{v}$
  (i.e., along reference direction i);
  e.g., v_x is the x-component of $\vec{v}$ in 2D

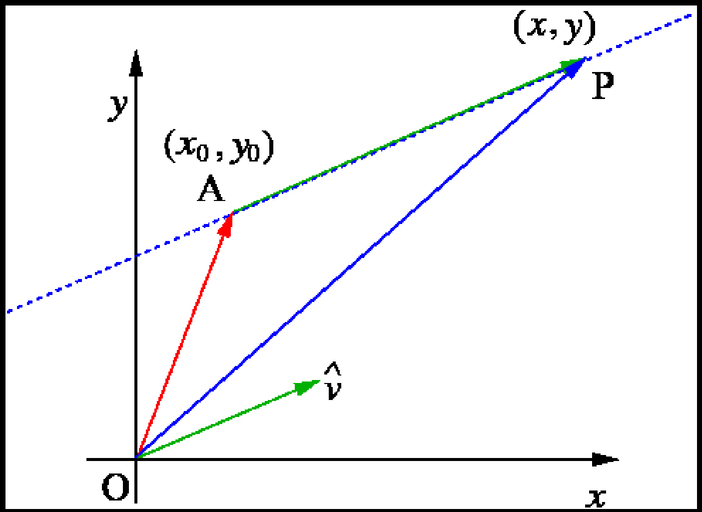


Lines in 2D

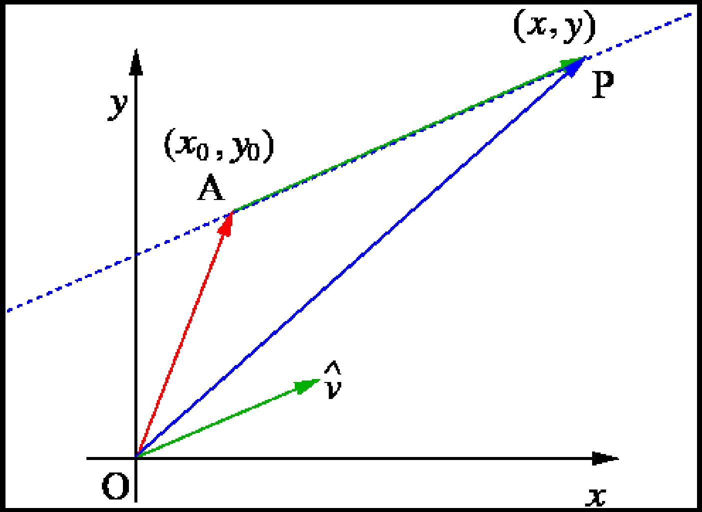
Shooting rays for a line in 2D

- Given: from point A (x_0, y_0), a ray is shot in direction $\hat{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

  Q. Equation of point P (x, y) after the ray travels a distance $l > 0$? (using vector algebra)



Equation of a line in 2D



- $\overrightarrow{OA} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \overrightarrow{OA} + l\hat{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

  (parametric form; l as the length parameter)

  you'll really use this form a lot for shooting rays!

Equation of a line in 2D

- $\overrightarrow{OA} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, $\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \overrightarrow{OA} + l\hat{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

  (parametric form; l as the length parameter)

  $\Rightarrow \;\; x = x_0 + lv_x$; i.e., $l = \dfrac{(x - x_0)}{v_x}$. Also, $y = y_0 + lv_y = y_0 + \dfrac{v_y}{v_x}(x - x_0)$

Equation of a line in 2D

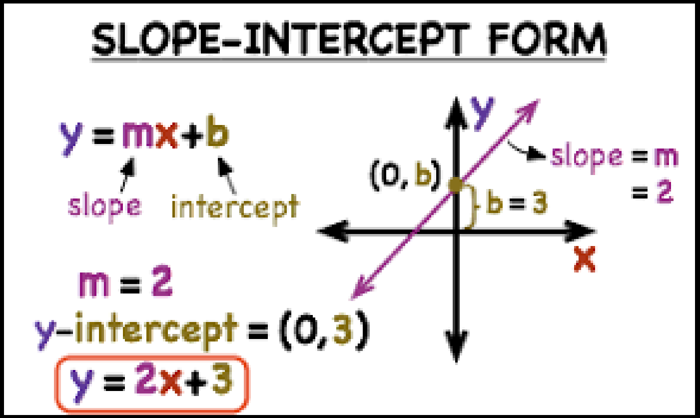
- $\overrightarrow{OA} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \overrightarrow{OA} + l\hat{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

  (parametric form; l as the length parameter)

  $\Rightarrow \; x = x_0 + l v_x; \; y = y_0 + l v_y = y_0 + \frac{v_y}{v_x}(x - x_0)$

- $y = \underbrace{\frac{v_y}{v_x}}_{\text{slope } m} x + \underbrace{\left(y_0 - \frac{v_y}{v_x} x_0\right)}_{\text{intercept } b} = mx + b$

  (slope-intercept form)



you probably have used the slope-intercept form in high school

Equation of a line in 2D

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- $y = \underbrace{\frac{v_y}{v_x}}_{\text{slope } m} x + \underbrace{\left(y_0 - \frac{v_y}{v_x} x_0 \right)}_{\text{intercept } b} = mx + b$ (slope-intercept form)

- Also, $Ax + By + C = 0$ (implicit form)

  the implicit form is useful for calculating distances, perpendiculars etc.

Line in 2D equations: solving a problem

- A ray shot in direction $\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ from point $(1, 2)$ yields a line L

  Q. (a) Find the equation of the line L in all forms

  (b) Find the equation of the line $\perp$ to L at $(4, 6)$

Line in 2D equations: solving a problem

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Q. (a) Find the equation of the line L in all forms

(b) Find the equation of the line $\perp$ to L at $(4, 6)$

A. (a.1) $x - 1 = 3l/5$, $y - 2 = 4l/5$; $l = \dfrac{5(x-1)}{3} = \dfrac{5(y-2)}{4}$

(a.2) $y = 4x/3 + 2/3$

(a.3) $4x - 3y + 2 = 0$

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(b.1) $\hat{v} \perp$ to L is $\pm \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$; $x - 4 = 4t/5,\ y - 6 = -3t/5$

(b.2) $y = -3x/4 + 9$

(b.3) $3x + 4y - 36 = 0$

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(b.2) $y = -3x/4 + 9$

(b.3) $3x + 4y - 36 = 0$ Vector $\begin{bmatrix} A \\ B \end{bmatrix}$ is $\perp$ to line $Ax + By + C = 0$ — this is one reason why the implicit form is so useful!

Equation of a line in 2D

- $\overrightarrow{OA} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix} = \overrightarrow{OA} + l\hat{v} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
  (parametric form; l as the length parameter)
- $y = \underbrace{\frac{v_y}{v_x}}_{\text{slope } m} x + \underbrace{\left(y_0 - \frac{v_y}{v_x} x_0 \right)}_{\text{intercept } b} = mx + b$ (slope-intercept form)
- $Ax + By + C = 0$ (implicit form)
- $m = \frac{v_y}{v_x} = -\frac{A}{B}, b = -\frac{C}{B}, v_x = \frac{1}{\sqrt{1+m^2}}, v_y = \frac{m}{\sqrt{1+m^2}},$
  $\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \pm \frac{1}{\sqrt{A^2+B^2}} \begin{bmatrix} A \\ -B \end{bmatrix}$, vector $\begin{bmatrix} A \\ B \end{bmatrix}$ is $\perp$ to line $Ax+By+C=0$,
  (shortest) distance from the origin to line $Ax+By+C=0$ is $\frac{|C|}{\sqrt{A^2+B^2}}$

Equation of a line in 2D

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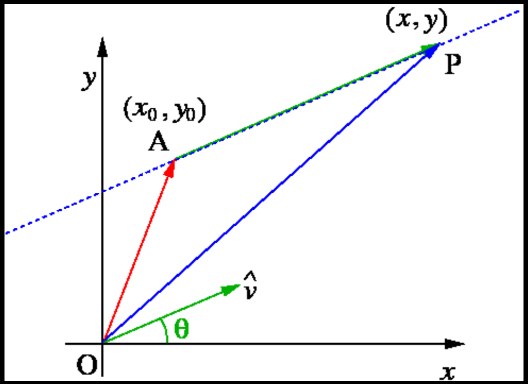
- $y = \underbrace{\frac{v_y}{v_x}}_{\text{slope } m} x + \underbrace{\left[y_0 - \frac{v_y}{v_x} x_0\right]}_{\text{intercept } b} = mx + b$ (slope-intercept form)
- $m = \frac{v_y}{v_x} = -\frac{A}{B}, b = -\frac{C}{B}; Ax + By + C = 0$ (implicit form)

Q. m in terms of θ?



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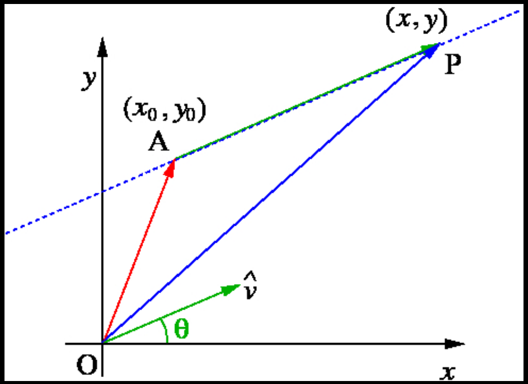
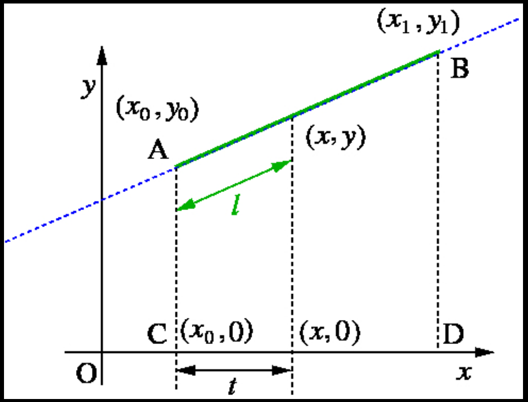
A. $m = \tan\theta$



Projections and shadows

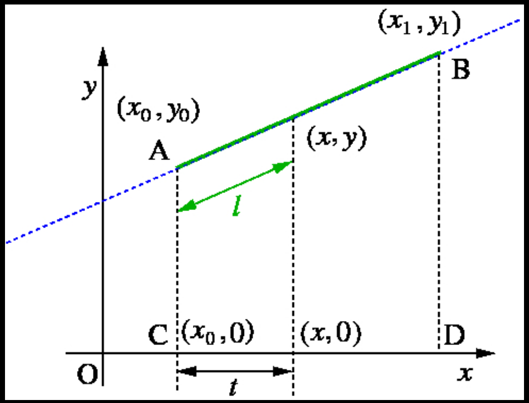
Projection in 2D on the x-axis



- Given: (x_0, y_0) and (x_1, y_1)

  Q. Express t as a function of l

Projection in 2D on the x-axis



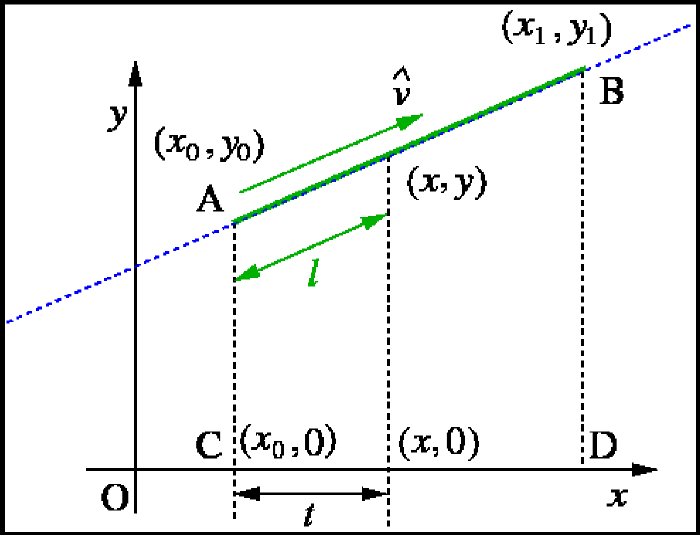
- Given: (x_0, y_0) and (x_1, y_1)

  Q. Express t as a function of l

  A. Line CD: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} + t \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{x}} \quad \Rightarrow \quad t = x - x_0$

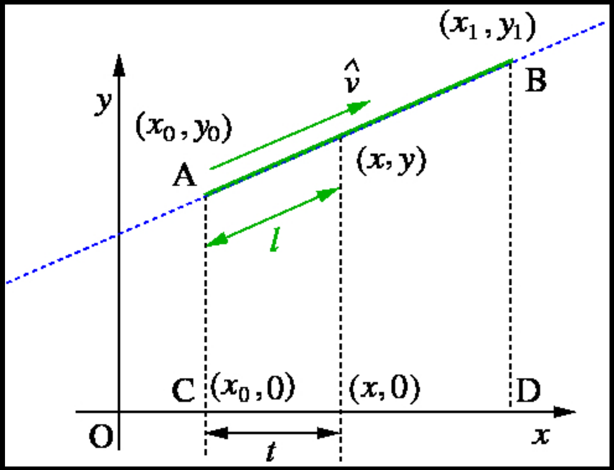
  Q′. How does one determine x?

Projection in 2D on the x-axis



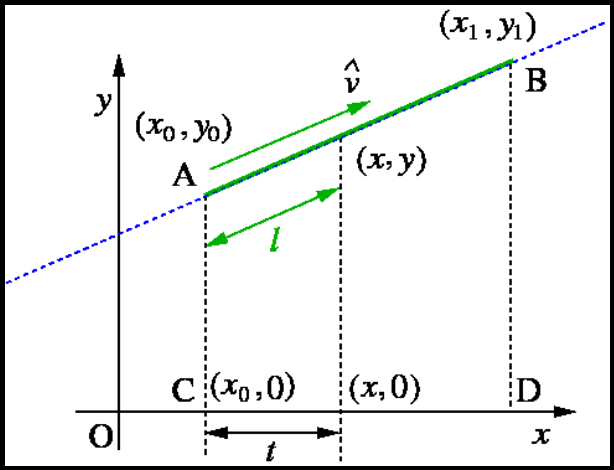
What is $\hat{v}$?

Projection in 2D on the x-axis



$$\hat{v} = \frac{1}{\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}} \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$$

Projection in 2D on the x-axis



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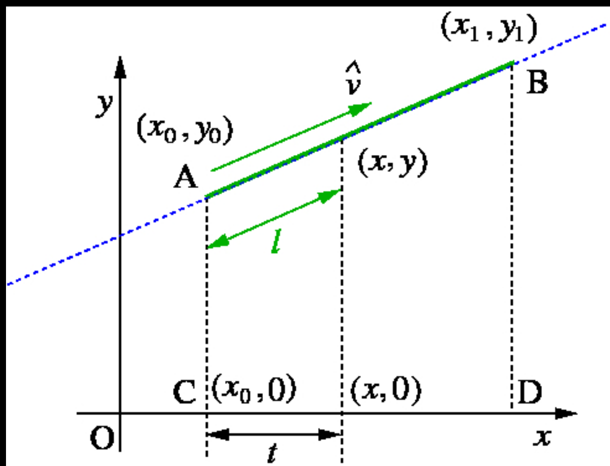
- Line AB: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix} \quad \Rightarrow \quad x = x_0 + l v_x$
- $t = x - x_0 = l v_x$

Projection in 2D on the x-axis



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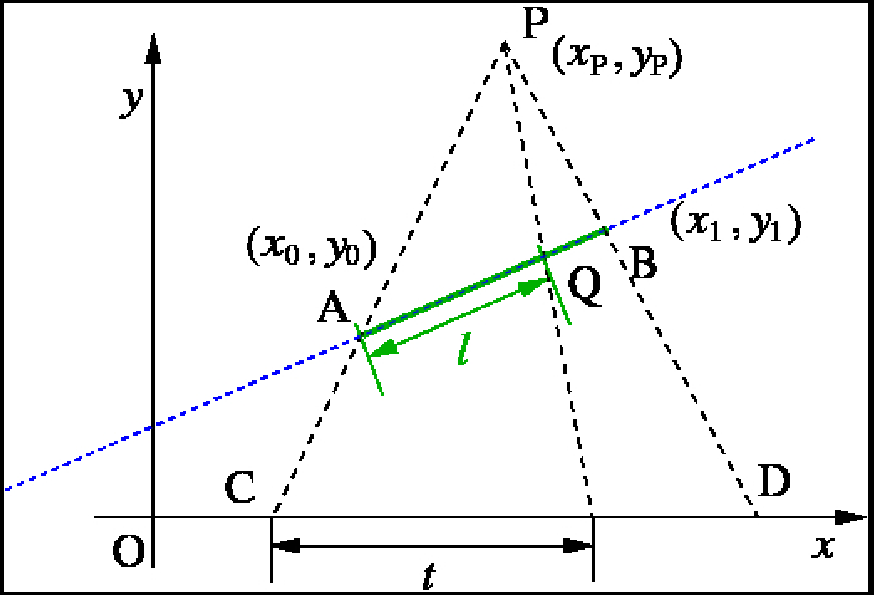
- $t = l v_x$; $t = l(\hat{v} \cdot \hat{x})$ (think components!)

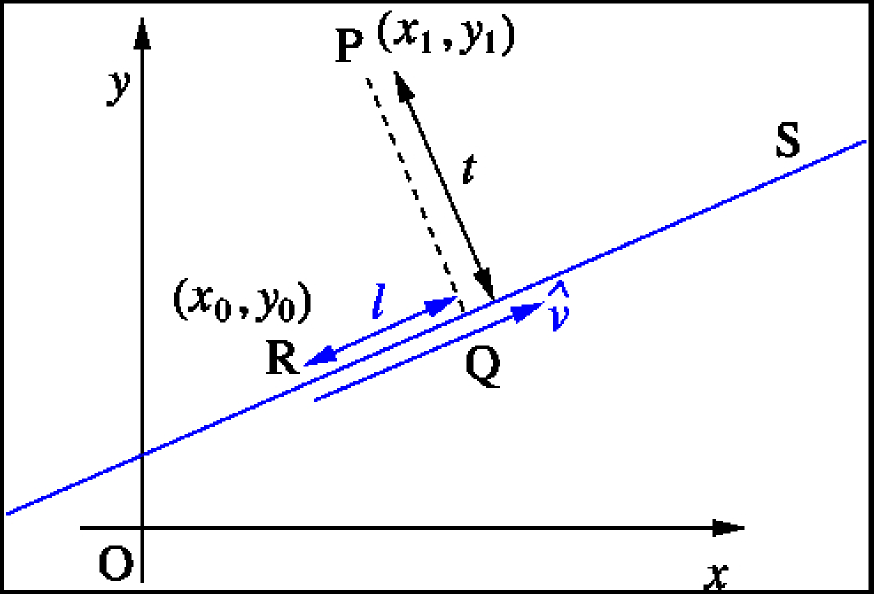
Shadows in 2D on the x-axis: see exercise



- Given: (x_0, y_0), (x_1, y_1) and $(x_\mathrm{P}, y_\mathrm{P})$

  Q. The locations of C and D, and t as a function of l?

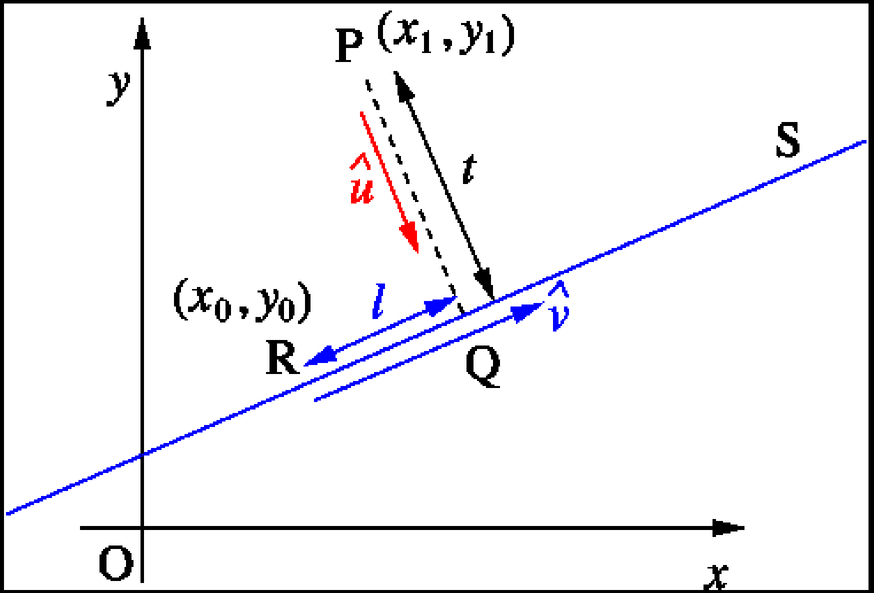
Projection of a point on a line



- Given: (x_0, y_0), (x_1, y_1) and $\hat{v}$, PQ $\perp$ RS

  Q. Determine the location of Q, and determine t and l

Projection of a point on a line

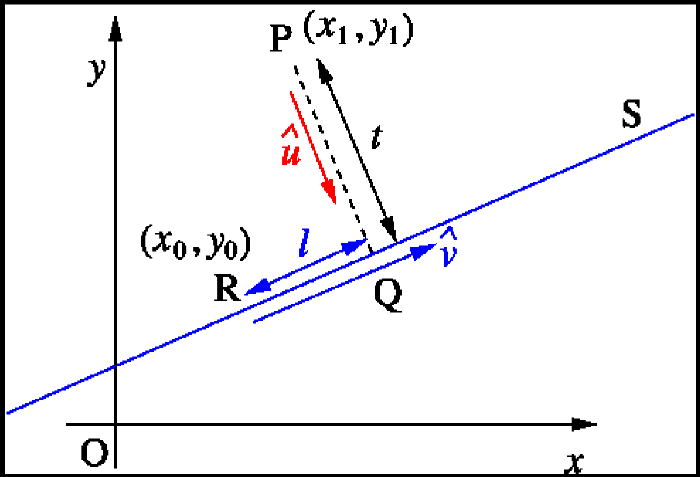


- Given: (x_0, y_0), (x_1, y_1) and $\hat{v}$, PQ $\perp$

  Q. Determine the location of Q, and determine t and l

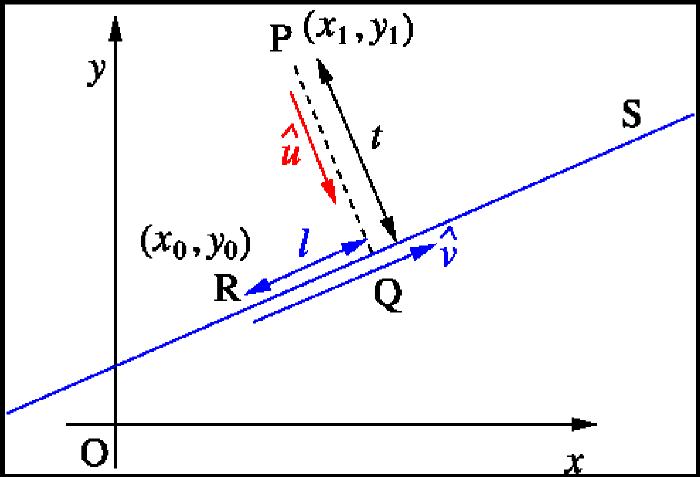
  A. Determine $\hat{u} \perp$ to RS, shoot a ray from P, intersecting RS at Q

Projection of a point on a line



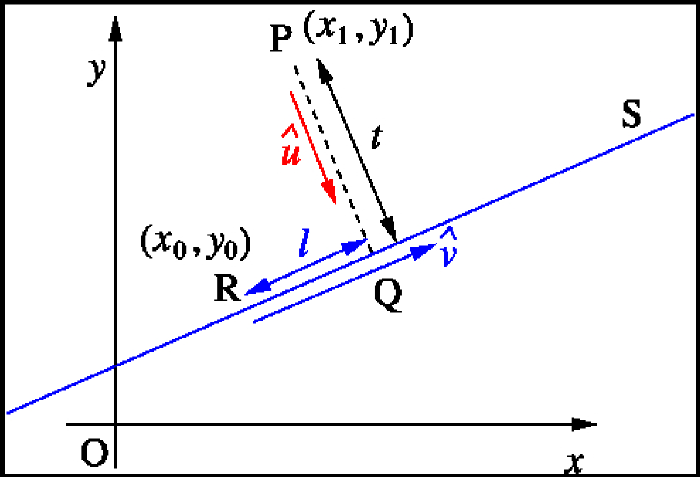
What is $\hat{u}$?

Projection of a point on a line



$$\hat{u} = \pm \begin{bmatrix} v_y \\ -v_x \end{bmatrix}$$

Projection of a point on a line



$$\hat{u} = \pm \begin{bmatrix} v_y \\ -v_x \end{bmatrix}$$

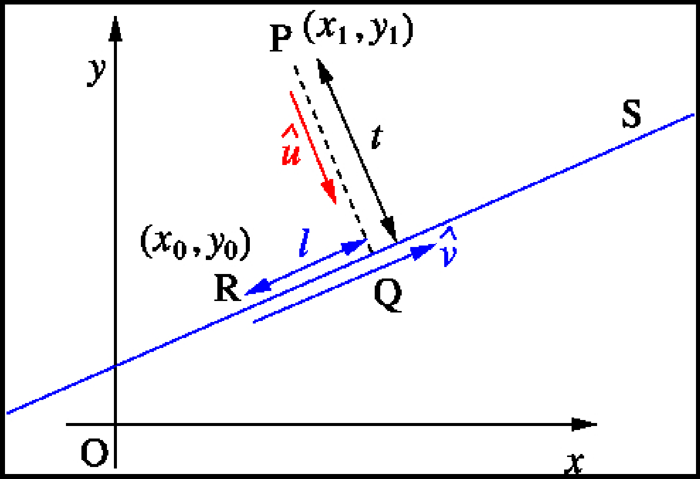
$$\frac{u_y}{u_x} = \frac{B}{A}$$

(recall implicit form

RS: $Ax + By + C = 0$)

$$\Rightarrow \hat{u} \; || \; \pm \begin{bmatrix} A \\ B \end{bmatrix}$$

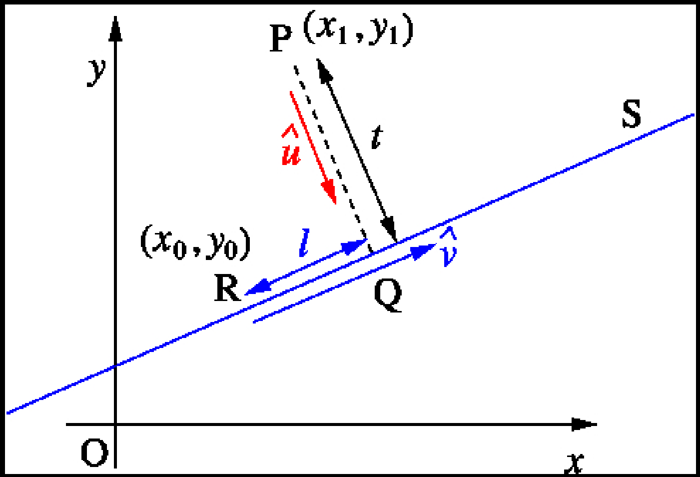
Projection of a point on a line



$$\hat{u} = \begin{bmatrix} -v_y \\ v_x \end{bmatrix}$$

- $\hat{v}$ is the tangent vector for the line, and $\pm\hat{u}$ is the normal vector

Projection of a point on a line

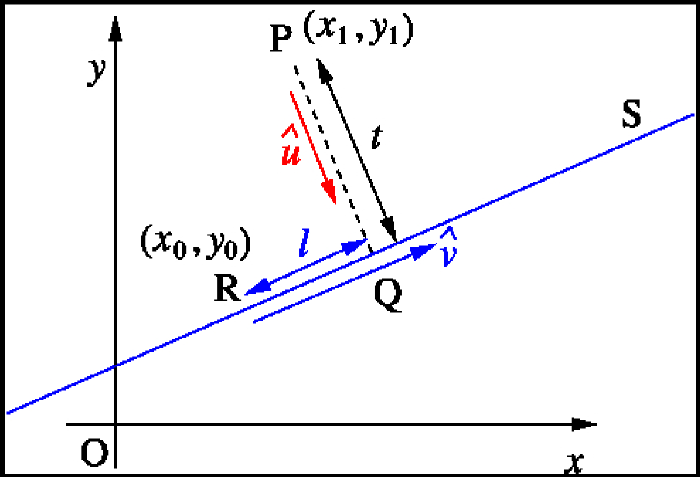


$$\hat{u} = \begin{bmatrix} v_y \\ -v_x \end{bmatrix}$$

Shoot a ray from P, $\perp$ AB

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\hat{u}$$

Projection of a point on a line



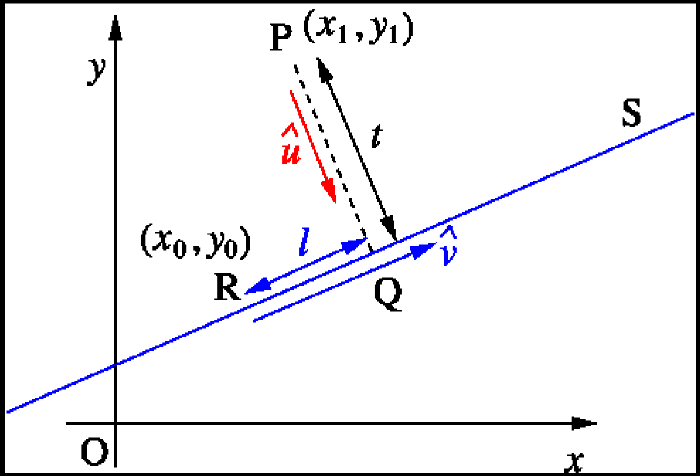
$$\hat{u} = \begin{bmatrix} v_y \\ -v_x \end{bmatrix}$$

Shoot a ray from P, ⊥ AB

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\hat{u}$$

also, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l\hat{v}$

Projection of a point on a line



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Shoot a ray from P, $\perp$ AB

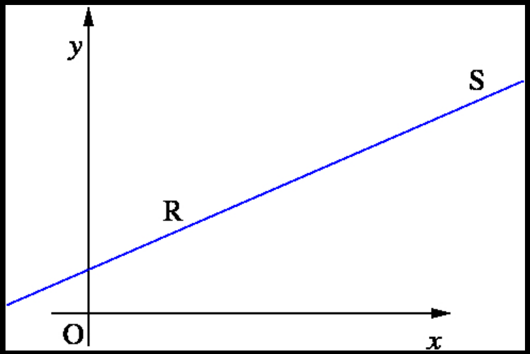
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + t\hat{u}$$

also, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l\hat{v}$

- $x_Q = x_0 + lv_x = x_1 - tv_y$; $y_Q = y_0 + lv_y = y_1 + tv_x$; solve l and t

A line $f(x, y) = y - (mx + b) = 0$ divides the plane in two

- On one side $f(x, y) > 0$, and on the other $f(x, y) < 0$

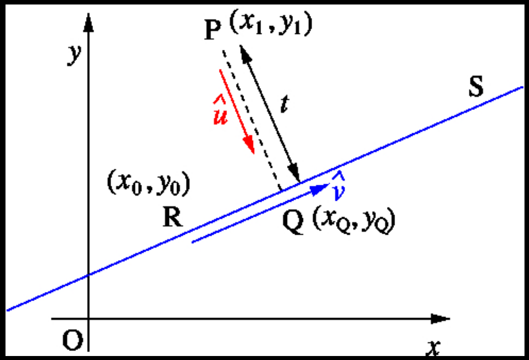


A line $f(x, y) = y - (mx + b) = 0$ divides the plane in two

- On one side $f(x, y) > 0$, and on the other $f(x, y) < 0$

RS: $f(x, y) = y - \left[\frac{v_y}{v_x}x + y_0 - \frac{v_y}{v_x}x_0\right] = 0$

$$f(x_Q, y_Q) = 0 \quad \Rightarrow \quad y_Q = \frac{v_y}{v_x}x_Q + y_0 - \frac{v_y}{v_x}x_0$$



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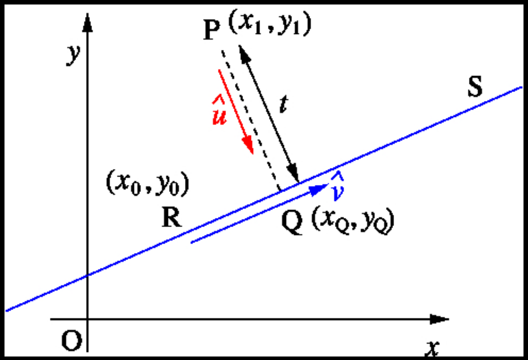

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$$f(x_1, y_1) = \underbrace{(y_1 - y_Q)}_{=-tu_y} - \frac{v_y}{v_x}\underbrace{(x_1 - x_Q)}_{=-tu_x} = t\,\frac{u_x v_y - u_y v_x}{v_x}$$

A line $f(x,y) = y - (mx+b) = 0$ divides the plane in two

- On one side $f(x,y) > 0$, and on the other $f(x,y) < 0$

RS: $f(x,y) = y - \left[\frac{v_y}{v_x}x + y_0 - \frac{v_y}{v_x}x_0\right] = 0$

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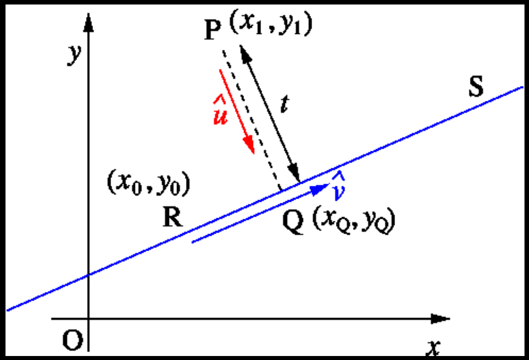

$$f(x_1, y_1) = \underbrace{(y_1 - y_Q)}_{=-tu_y} - \frac{v_y}{v_x}\underbrace{(x_1 - x_Q)}_{=-tu_x} = t\,\frac{u_x v_y - u_y v_x}{v_x}$$

if P lies on the other side of line RS then

$$f(x_1, y_1) = -t\,\frac{u_x v_y - u_y v_x}{v_x}$$

Summary: Vector algebra, and uses in 2D

- Dot product of vectors
- Shooting rays for a line
  - equations: parametric, slope-intercept and implicit forms
  - parallel and perspective projections of a line
  - projection of a point on a line, tangent and normal vectors

Summary: vector algebra, and uses in 2D

- Dot product of vectors
- Shooting rays (along a line)
  - equations: parametric, slope-intercept and implicit forms
  - parallel and perspective projections of a line
  - projection of a point on a line, tangent and normal vectors
- Next class: circles, ellipses, lines in 3D, (hyper)planes and cross product

Finally, references

- Book chapter 2: Miscellaneous Math
  - Sec. 2.5, only the relevant parts for 2D