Graphics (INFOGR), 2018-19, Block IV, lecture 3

Deb Panja

Today: Circles, ellipses, lines in 3D, (hyper)planes and cross product

Welcome

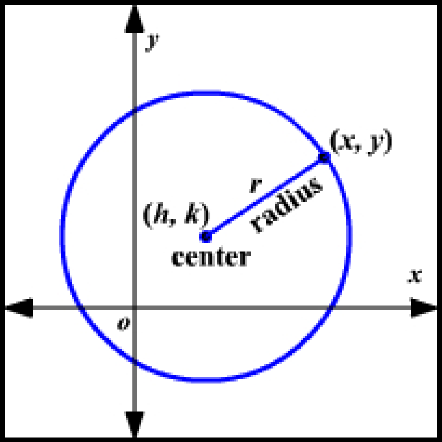
Today

- Circles, ellipses and shooting rays
- Lines and planes in 3D
- Cross products of vectors
- (Hyper)planes and normals

Circles, ellipses and shooting rays

Equations of circles

- Circle: $(x-h)^2+(y-k)^2=r^2$
  (h,k): location of the centre



Equations of circles and ellipses

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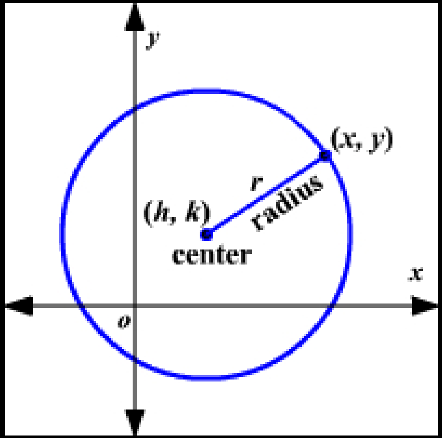
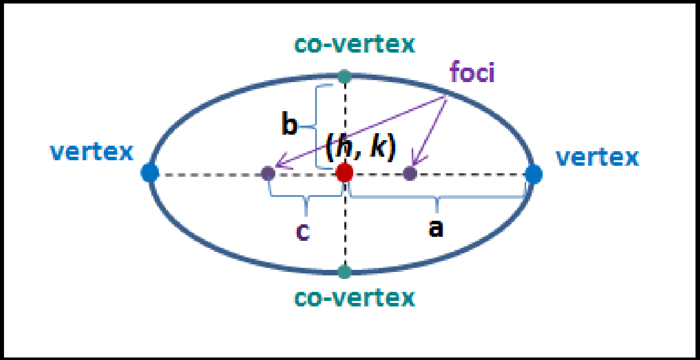


- Ellipse: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

  (h, k): location of the centre

  a: semi-major axis, b: semi-minor axis



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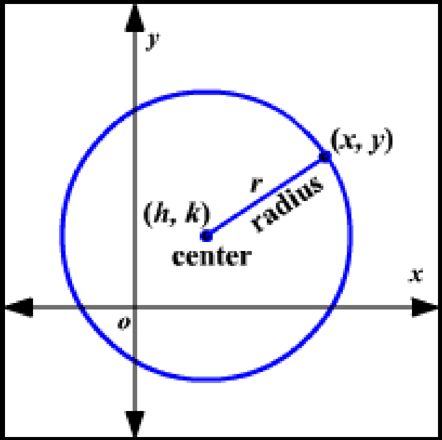

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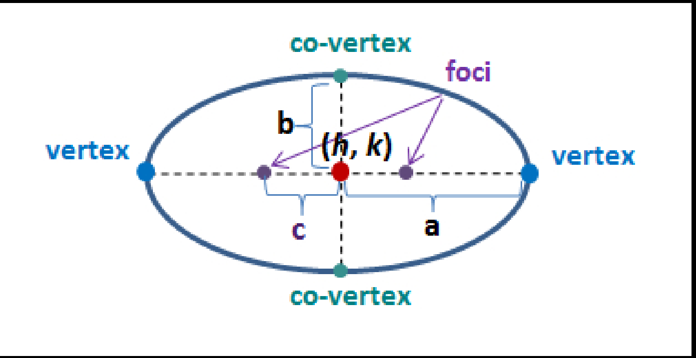
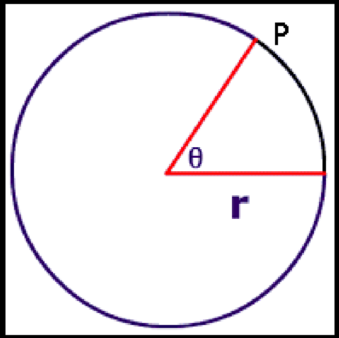
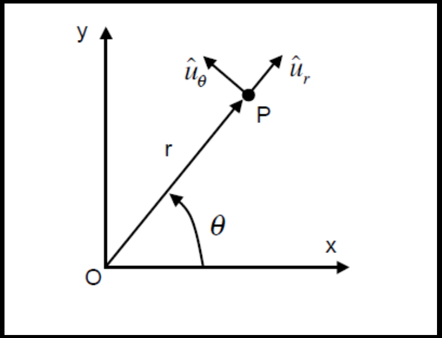


(both of the above are implicit forms)

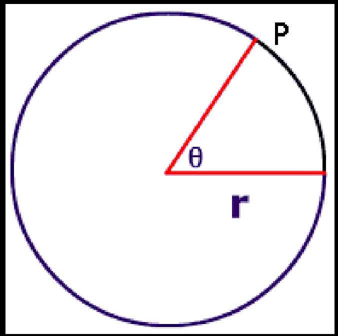
Circular co-ordinate system

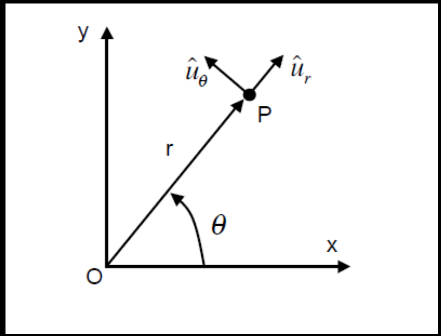




- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix} + r\,\hat{u}_r; \quad \hat{u}_r = ?$

Circular co-ordinate system





- $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix} + r\,\hat{u}_r; \quad \hat{u}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}; \quad \hat{u}_\theta = ?$

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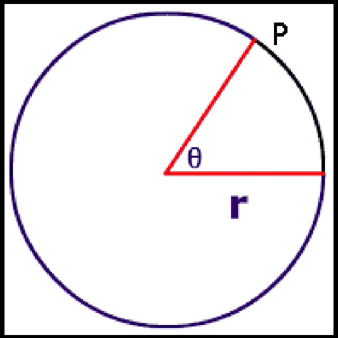

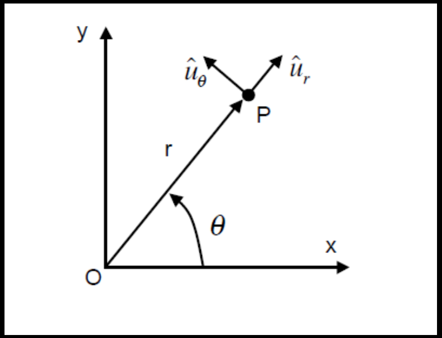

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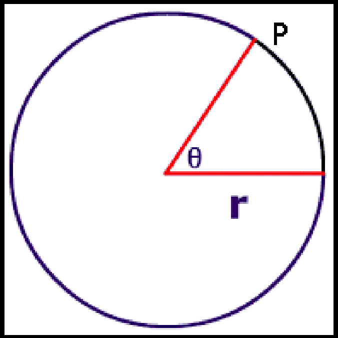

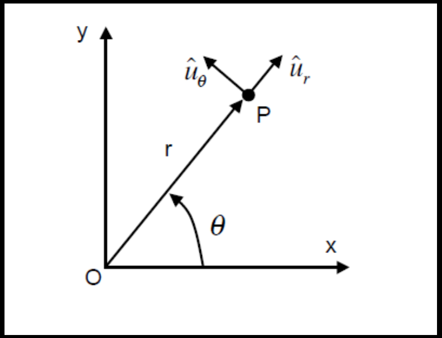

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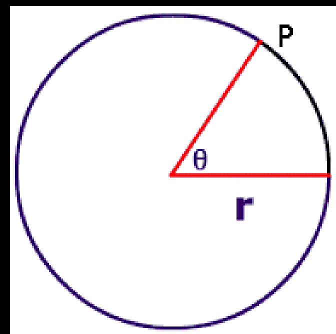
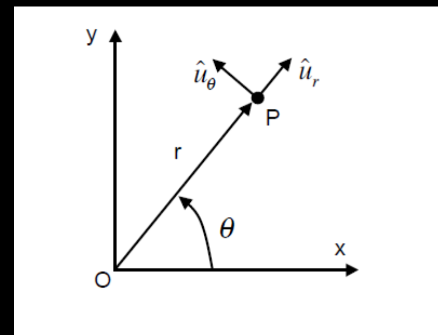
- Given (h, k) and r, P is described by θ alone:
  $x = h + r\cos\theta$, $y = k + r\sin\theta$ (parametric form)

The circular co-ordinate system





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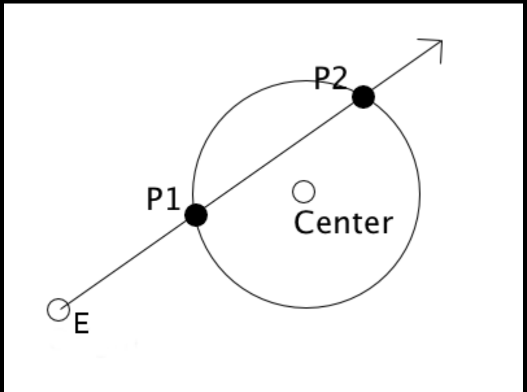
- Given (h, k) and r, P is described by θ alone:
  $x = h + r\cos\theta$, $y = k + r\sin\theta$ (parametric form)
- $\hat{u}_r$ unit normal vector; $\hat{u}_\theta$ unit tangent vector to the circle at P

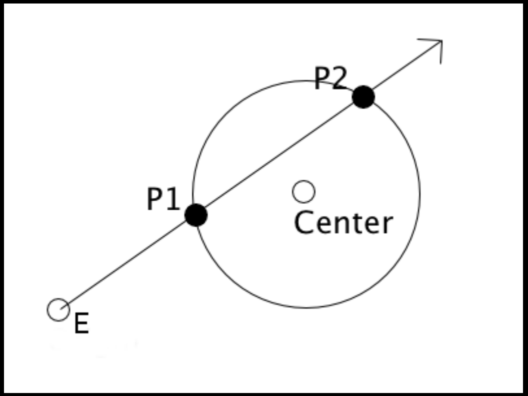
Shooting rays at a circle



- Given: Eye at (x_0, y_0), ray along $\hat{v}$, circle centre at (h, k), radius r

  Q. Find locations of P1 and P2

Shooting rays at a circle

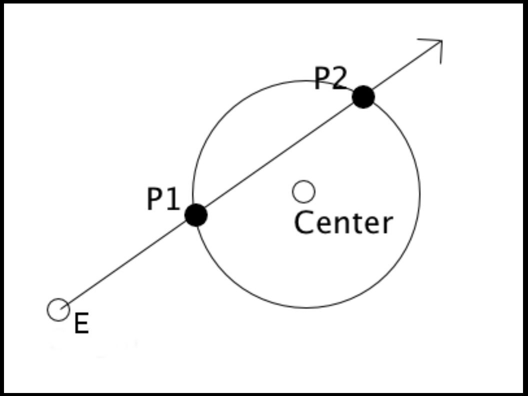


- Given: Eye at (x_0, y_0), ray along $\hat{v}$, circle centre at (h, k), radius r

  Q. Find locations of P1 and P2

  A. Shoot a ray from E along $\hat{v}$, intersect circle at P1 and P2

Shooting rays at a circle



- Given: E at (x_0, y_0), ray along $\hat{v}$, circle centre at (h, k), radius r

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \end{bmatrix}; \qquad (x-h)^2 + (y-k)^2 = r^2$$

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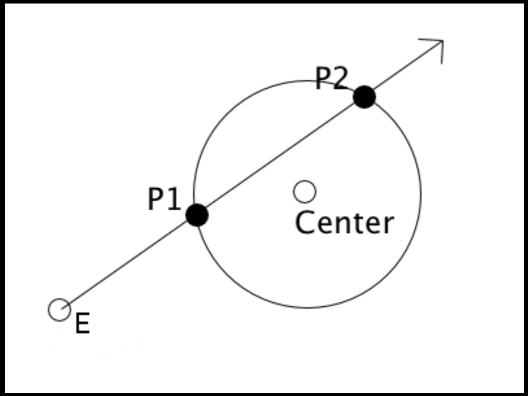

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$\Rightarrow$ quadratic equation in l: $al^2 + bl + c = 0$

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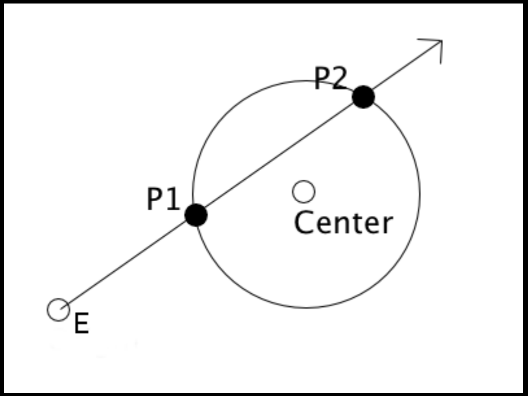

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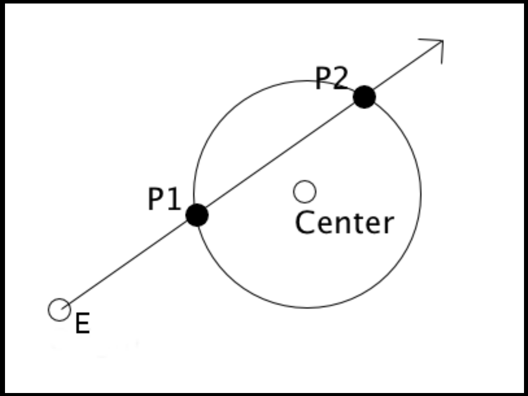
$\Rightarrow$ quadratic equation in l: $al^2 + bl + c = 0$

$$\Rightarrow l = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Shooting rays at a circle



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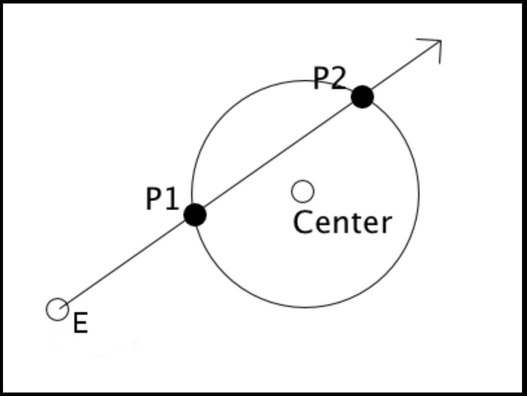
$\Rightarrow$ quadratic equation in l: $al^2 + bl + c = 0 \quad \Rightarrow \quad l = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ (i) $b^2 - 4ac > 0$: (P1,P2); (ii) $b^2 - 4ac = 0$: P1 $=$ P2 (tangent ray); (iii) $b^2 - 4ac < 0$: no intersection; $\Rightarrow$ (θ_1, θ_2)-values for P1 and P2

Shooting rays at a circle



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  $\Rightarrow$ (i) $b^2 - 4ac > 0$: (P1,P2); (ii) $b^2 - 4ac = 0$: P1 = P2 (tangent ray);
  (iii) $b^2 - 4ac < 0$: no intersection; $\Rightarrow$ (θ_1, θ_2)-values for P1 and P2

  P2 will not be visible to the eye!

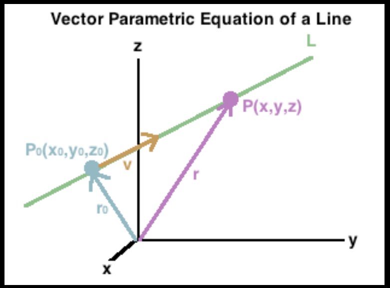
Ingredients for 3D, and recap

- Will work in 3D, Cartesian co-ordinates: (x, y, z) reference directions
  - in Cartesian co-ordinates $\hat{x}$, $\hat{y}$ and $\hat{z}$ as basis vectors
- A point P: (x, y, z), a vector $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$
  - will use vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to reach point (x, y, z) from the origin

Lines and planes in 3D

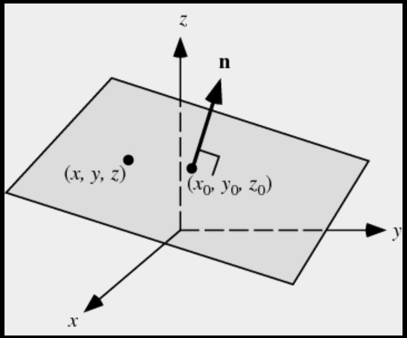
Shooting a ray for a line in 3D

- $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + l\hat{v} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + l \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$
  (parametric form)



- $$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z} \quad (= l)$$ [equivalent slope-intercept form]

A plane in 3D

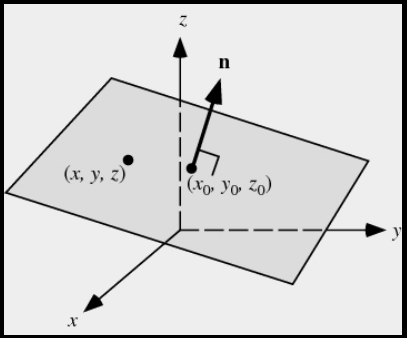


Given: $\hat{n}$, normal to the plane, and (x_0, y_0, z_0) on the plane

Q. What is the equation of this plane?

(uses: storing a plane in the memory, distance to the plane...)

A plane in 3D

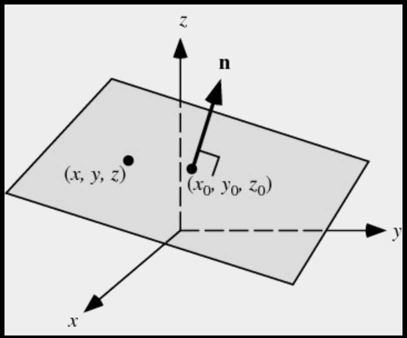


Given: $\hat{n}$, normal to the plane, and (x_0, y_0, z_0) on the plane

Q. What is the equation of this plane?

A. Plane equation: $(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0$

A plane in 3D

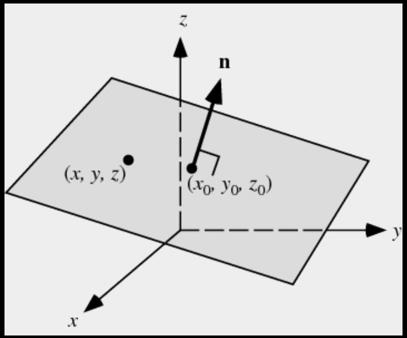


- Plane equation: $(x - x_0)n_x + (y - y_0)n_y + (z - z_0)n_z = 0$

  $n_x x + n_y y + n_z z - (n_x x_0 + n_y y_0 + n_z z_0) = 0$, or

  $\underbrace{Ax + By + Cz + D = 0}_{\text{implicit form}}$ (recall line in 2D form $Ax + By + C = 0$)

A plane in 3D



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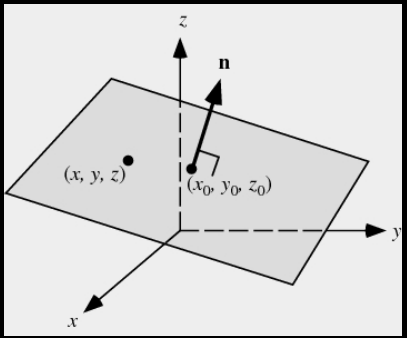
- A plane divides the 3D space into two:

  for one $Ax + By + Cz + D > 0$, for the other $Ax + By + Cz + D < 0$

A plane in 3D



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- A plane divides the 3D space into two:
  for one $Ax + By + Cz + D > 0$, for the other $Ax + By + Cz + D < 0$

- Similarly a hyperplane also divides the space into two

Cross products of vectors

Cross product of vectors

Q. Why?

Cross product of vectors

Q. Why?

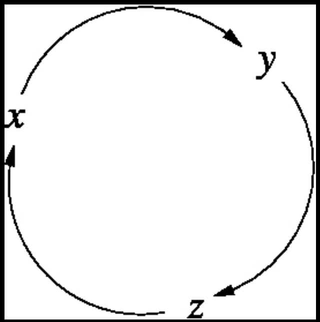
A. Because it allows us to easily switch between planes and their normals

Cross product of two vectors

- $\vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}; \quad \vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

  $\vec{w} = \vec{u} \times \vec{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$: a vector

  $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$



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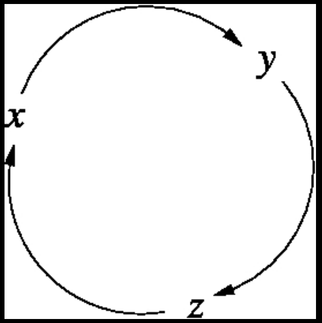

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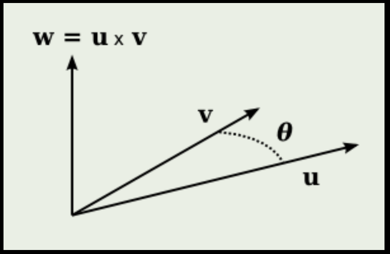
- Geometric interpretation:
  $\vec{w} \perp (\vec{u}, \vec{v})$; $\vec{w} \cdot \vec{u} = ?$



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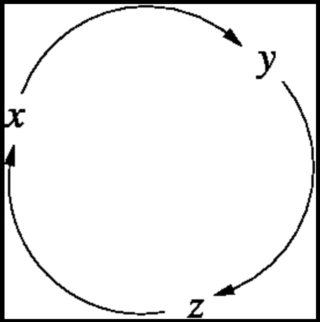
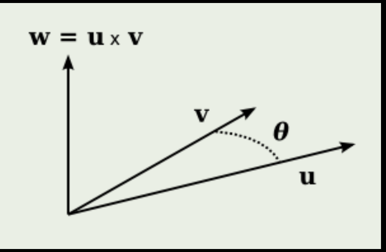


- Geometric interpretation:

  $\vec{w} \perp (\vec{u}, \vec{v})$; $\vec{w} \cdot \vec{u} = \vec{w} \cdot \vec{v} = 0$

  $||\vec{w}|| = ||\vec{u}|| \; ||\vec{v}|| \; \sin\theta$



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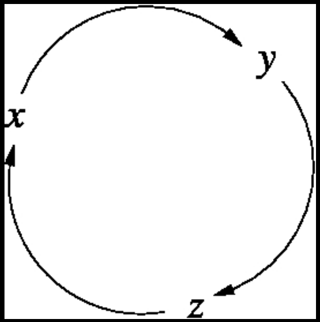

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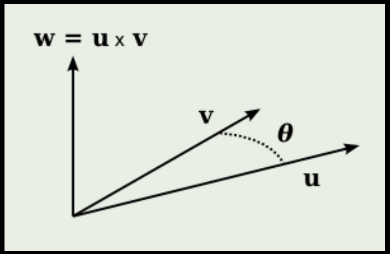

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  $\vec{u}$ and $\vec{v}$ are parallel; $\vec{w} = \vec{u} \times \vec{v} = ?$

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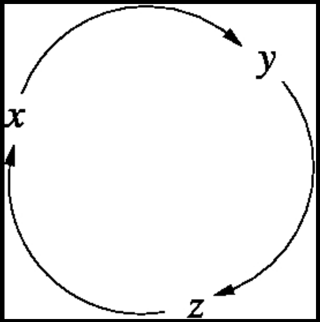

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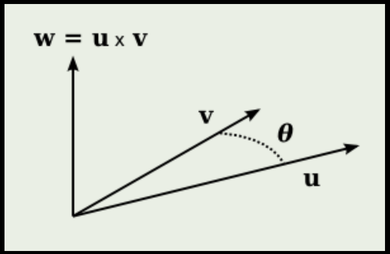
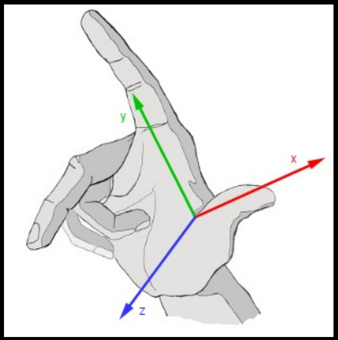


Cross product and handedness of a co-ordinate system

- Right-handed co-ordinate system: $\hat{x} \times \hat{y} = \hat{z}$, $\hat{y} \times \hat{z} = \hat{x}$, $\hat{z} \times \hat{x} = \hat{y}$

  i.e., $(\hat{x} \times \hat{y}) \cdot \hat{z} > 0$ etc.

  (this is what we will use)



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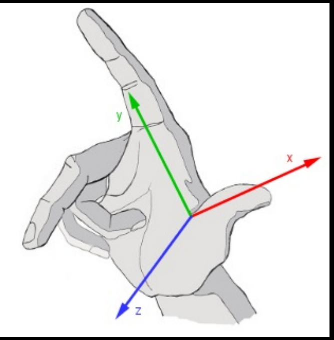

- Left-handed co-ordinate system: $\hat{x} \times \hat{y} = -\hat{z}$, $\hat{y} \times \hat{z} = -\hat{x}$, $\hat{z} \times \hat{x} = -\hat{y}$
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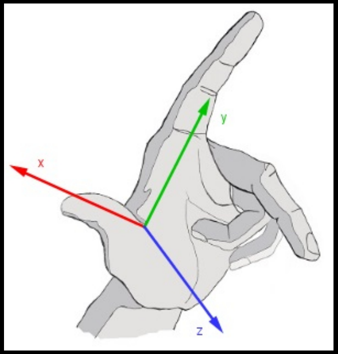


(Hyper)planes and normals

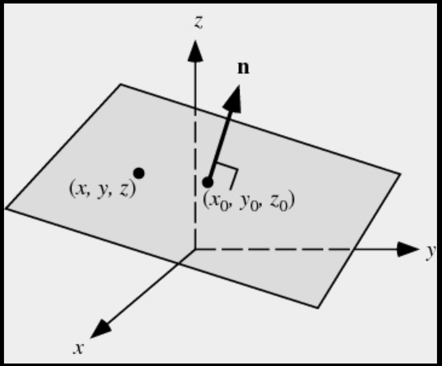
Revisit: (hyper)planes and normals

- Plane equation $Ax + By + Cz + D = 0$ (implicit form)

  like line in 2D, normal $\hat{n} \,||\, \pm \begin{bmatrix} A \\ B \\ C \end{bmatrix}$

  likewise, (shortest) distance from the origin is $\dfrac{|D|}{\sqrt{A^2 + B^2 + C^2}}$

  (similarly for hyperplanes)



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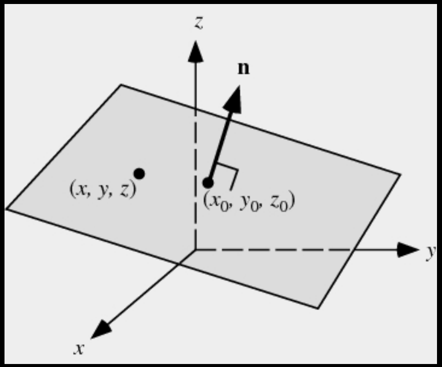
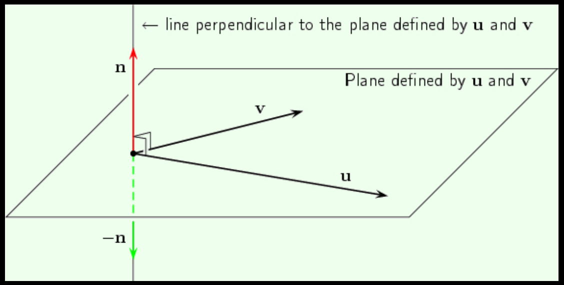


- Plane using two (planar) vectors (parametric form)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + l\hat{u} + t\hat{v}$$

  (similarly for hyperplanes)



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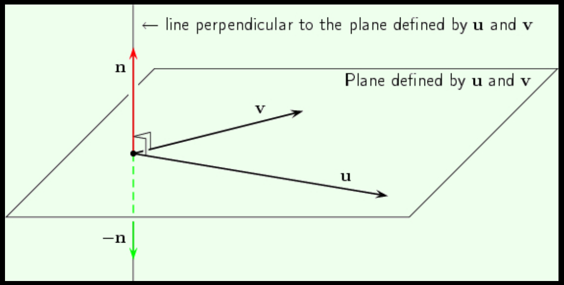

(similarly for hyperplanes)

$\hat{n} \,||\, \hat{u} \times \hat{v}$ Q. Given $\vec{u}$ and $\vec{v}$ on a plane can you calculate $\hat{n}$?

Revisit: (hyper)planes and normals

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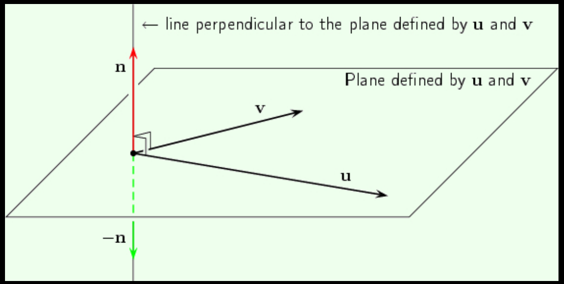

(similarly for hyperplanes)

$\hat{n} \mid\mid \hat{u} \times \hat{v}$ A. Given $\vec{u}$ and $\vec{v}$ on a plane $\hat{n} = \vec{u} \times \vec{v} / ||\vec{u} \times \vec{v}||$

for $\hat{u} \perp \hat{v}$, $(\hat{u}, \hat{v}, \hat{n})$ form a right-handed co-ordinate system

in that case, normal $\hat{n}$, tangent $\hat{u}$, bitangent $\hat{v}$

Summary: circles, ellipses, (hyper)planes and cross product

- Circles, ellipses and shooting rays at them
- Shooting rays as a line in 3D
  - equations: parametric and implicit-like forms
- Equation of a plane using the normal vector
- Cross product, left- and right-handed co-ordinate systems
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- Next class: primitives (continued) and projections in 3D

Finally, references

- Book chapter 2: Miscellaneous Math
  - Sec. 2.2, Sec. 2.4.4, 2.4.6-2.4.7
  - Sec. 2.5, the relevant parts for 3D