Graphics (INFOGR), 2018-19, Block IV, lecture 4

Deb Panja

Today: Primitives (continued) and projections in 3D

Welcome

Today

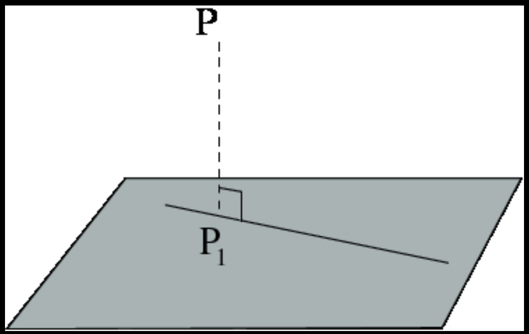
- Projections
- Lines and spheres in 3D

Recap, and ingredients

- Will work in 3D, Cartesian co-ordinates: (x, y, z) reference directions
  - in Cartesian co-ordinates $\hat{x}$, $\hat{y}$ and $\hat{z}$ as basis vectors
- A point P: (x, y, z), a vector $\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$
  - will use vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to reach point (x, y, z) from the origin

Projections in 3D

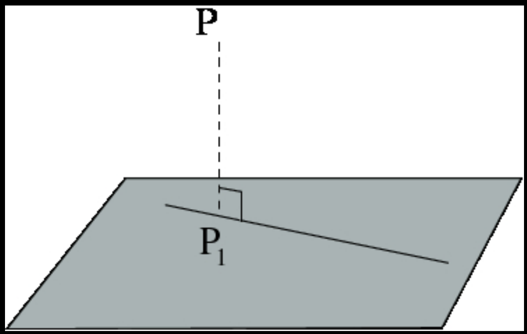
Projection of a point on a plane



- Given: P (x_0, y_0, z_0); plane $Ax + By + Cz + D = 0$

  Q. Location of P_1?

Projection of a point on a plane

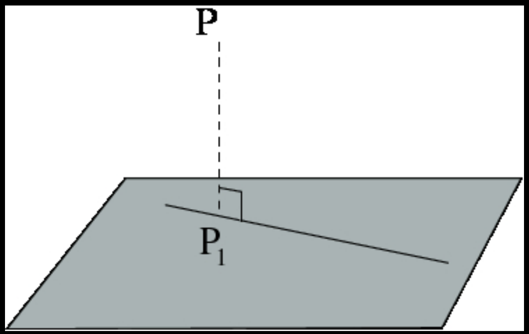


- Given: P (x_0, y_0, z_0); plane $Ax + By + Cz + D = 0$

  Q. Location of P_1?

  A. Shoot a ray from P $\perp$ to the plane, intersecting at P_1

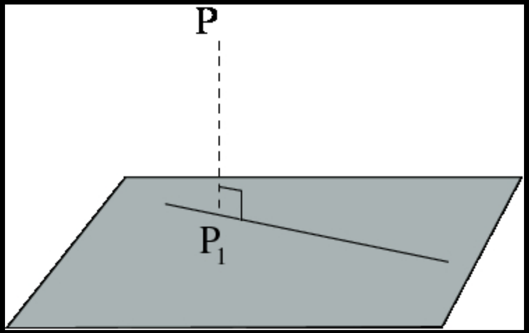
Projection of a point on a plane



- P: (x_0, y_0, z_0); plane: $Ax + By + Cz + D = 0$

  shoot a ray $\hat{v}$ from P $\perp$ to the plane

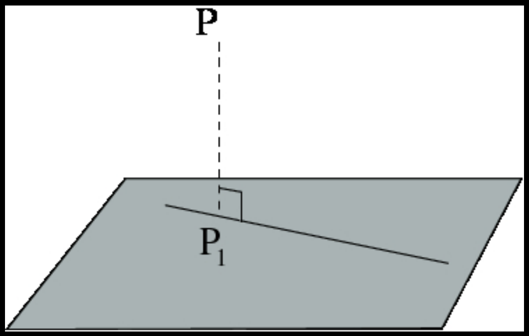
Projection of a point on a plane



- P: (x_0, y_0, z_0); plane: $Ax + By + Cz + D = 0$

$$\hat{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \pm \frac{1}{\sqrt{A^2 + B^2 + C^2}} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

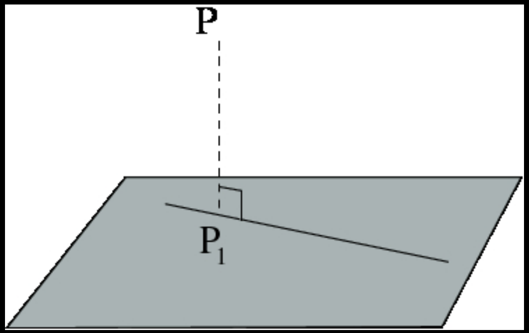
Projection of a point on a plane



- P: (x_0, y_0, z_0); plane: $Ax + By + Cz + D = 0$

$$\text{PP}_1\text{:} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + a \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Projection of a point on a plane

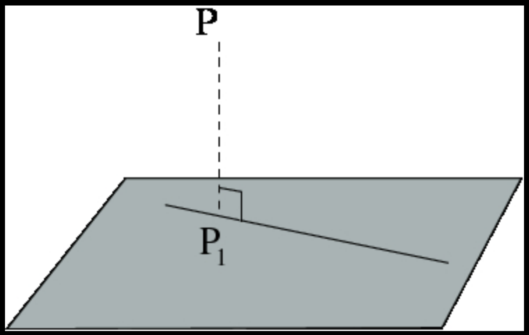


- P: (x_0, y_0, z_0); plane: $Ax + By + Cz + D = 0$

  P_1: $A(x_0 + av_x) + B(y_0 + av_y) + C(z_0 + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P_1

Projection of a point on a plane



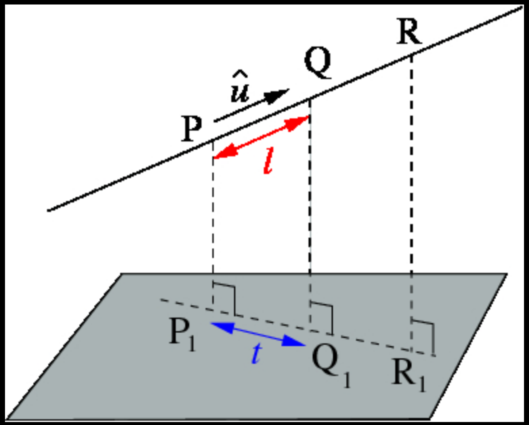
- P: (x_0, y_0, z_0); plane: $Ax + By + Cz + D = 0$

  P$_1$: $A(x_0 + av_x) + B(y_0 + av_y) + C(z_0 + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P$_1$

  (if $a < 0$ then use $-\hat{v}$ i.s.o. $\hat{v}$)

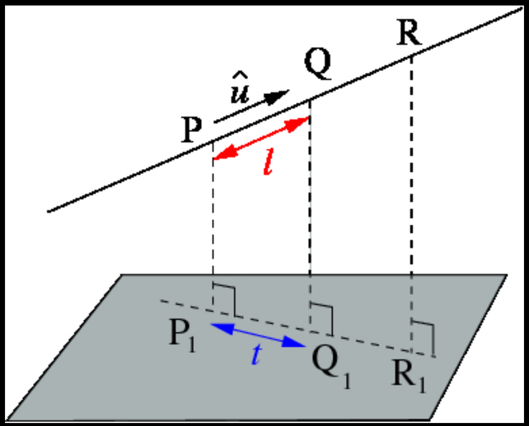
Projection of a line on a plane



- Given: P location (x_0, y_0, z_0), $\hat{u}$, plane equation $Ax + By + Cz + D = 0$

  Q. t as a function of l?

Projection of a line on a plane



- Given: P location (x_0, y_0, z_0), $\hat{u}$, plane equation $Ax + By + Cz + D = 0$

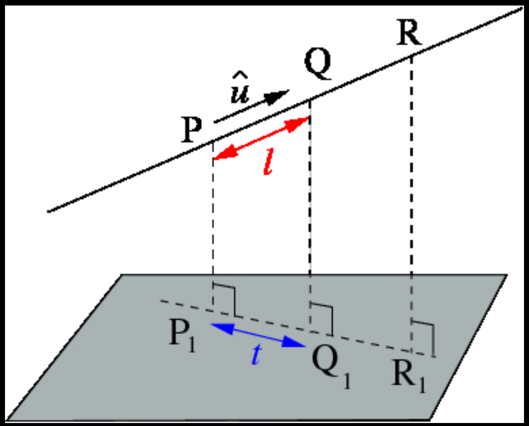
  Q. t as a function of l?

  A. Shoot a ray $\hat{v}$ from P, $\perp$ to the plane, intersecting at P_1

  Shoot a ray $\hat{v}$ from Q, $\perp$ to the plane, intersecting at Q_1

  Measure P_1Q_1, compare to PQ

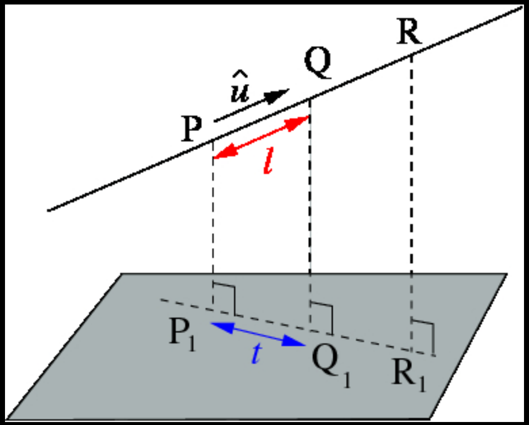
Projection of a line on a plane



- P: (x_0, y_0, z_0), $\hat{u}$, plane: $Ax + By + Cz + D = 0$

  shoot a ray $\hat{v}$ from P, $\perp$ plane

Projection of a line on a plane



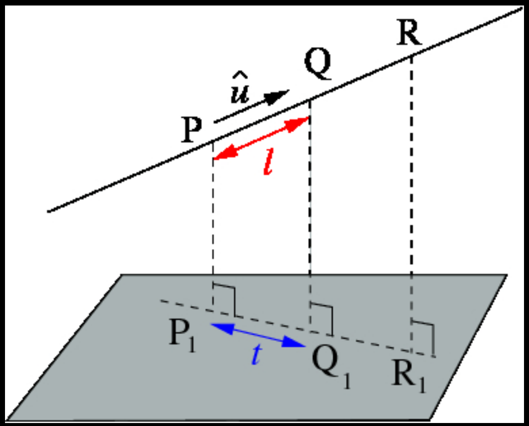
- P: (x_0, y_0, z_0), $\hat{u}$, plane: $Ax + By + Cz + D = 0$

  P$_1$: $A(x_0 + av_x) + B(y_0 + av_y) + C(z_0 + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P$_1$

  (if $a < 0$ then use $-\hat{v}$ i.s.o. $\hat{v}$)

Projection of a line on a plane



- P: (x_0, y_0, z_0), $\hat{u}$, plane: $Ax + By + Cz + D = 0$

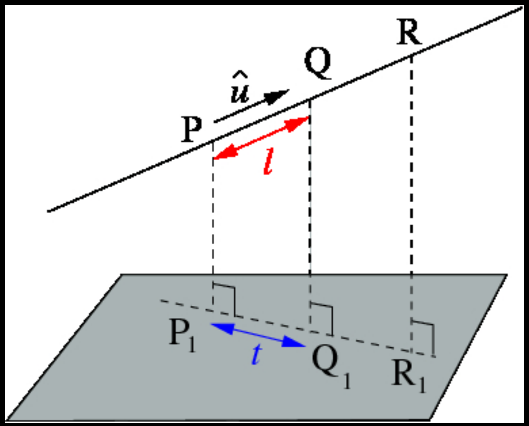
  P_1: $A(x_0 + av_x) + B(y_0 + av_y) + C(z_0 + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P_1

  Q_1: $A((x_0+lu_x)+bv_x)+B((y_0+lu_y)+bv_y)+C((z_0+lu_z)+bv_z)+D = 0$

  $\Rightarrow$ solve b; solve for the location of Q_1

Projection of a line on a plane



- P: (x_0, y_0, z_0), $\hat{u}$, plane: $Ax + By + Cz + D = 0$

  P_1: $A(x_0 + av_x) + B(y_0 + av_y) + C(z_0 + av_z) + D = 0$

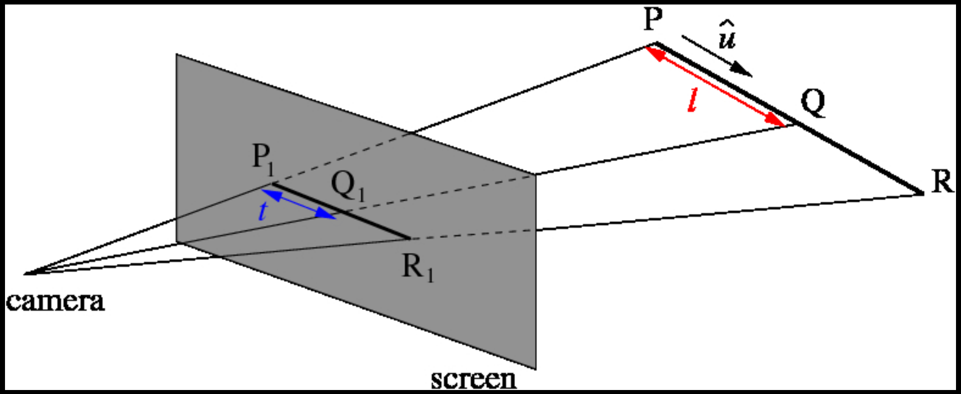
  $\Rightarrow$ solve a; solve for the location of P_1

  Q_1: $A((x_0+lu_x)+bv_x)+B((y_0+lu_y)+bv_y)+C((z_0+lu_z)+bv_z)+D = 0$

  $\Rightarrow$ solve b; solve for the location of Q_1

  determine t from the locations of P_1 and Q_1

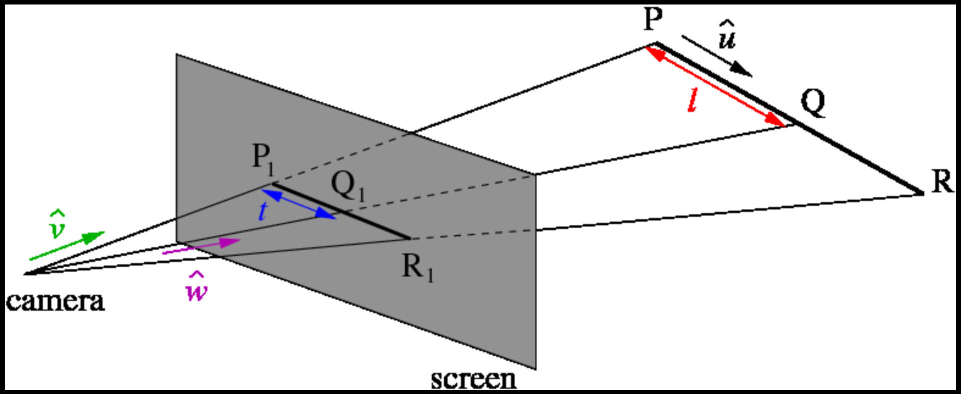
Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

  Q. Location of P_1, and t as a function of l?

Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

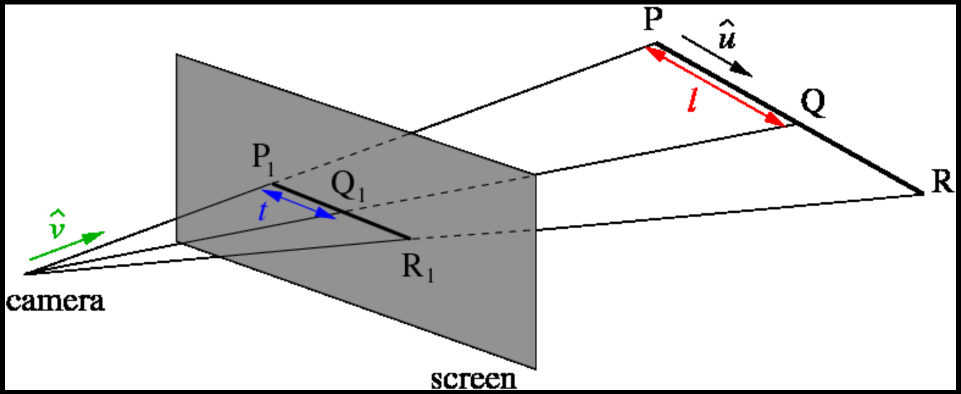
  Q. Location of P_1, and t as a function of l?

  A. Shoot a ray from camera along $\hat{v}$, intersecting screen at P_1

  Shoot a ray from camera along $\hat{w}$, intersecting screen at Q_1

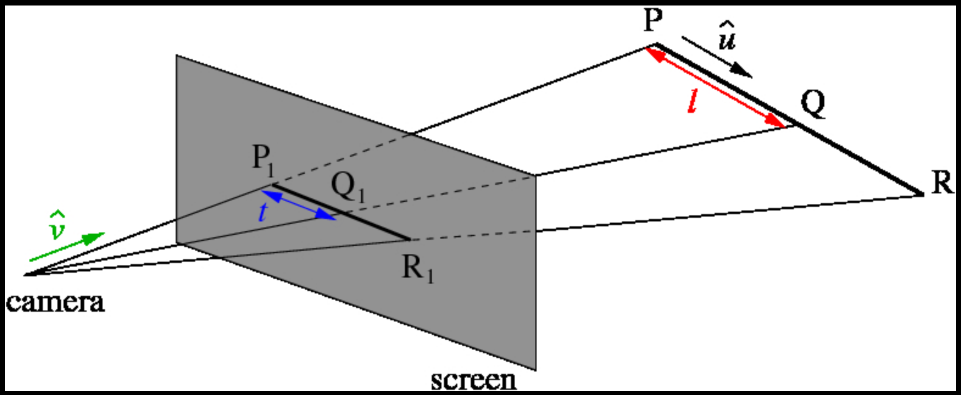
  Measure P_1Q_1, compare to PQ

Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$
  determine $\hat{v}$ from (x_0, y_0, z_0) and (x_c, y_c, z_c)

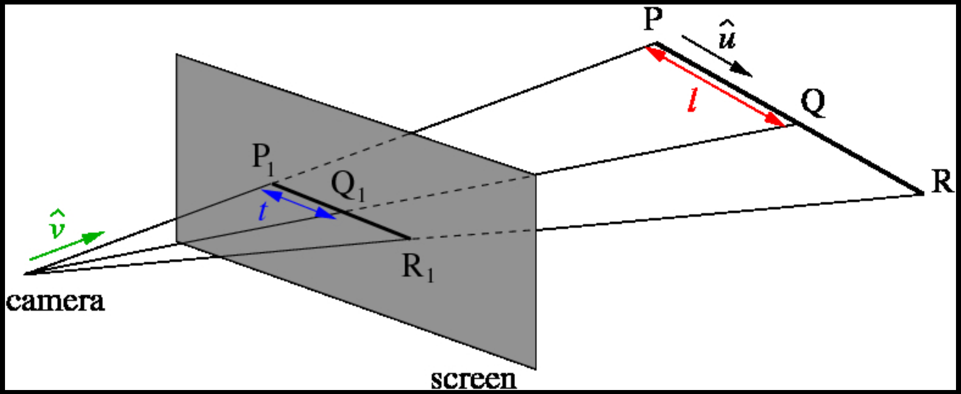
Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

$$\hat{v} = \frac{1}{\sqrt{(x_0 - x_c)^2 + (y_0 - y_c)^2 + (z_0 - z_c)^2}} \begin{bmatrix} x_0 - x_c \\ y_0 - y_c \\ z_0 - z_c \end{bmatrix}$$

Projecting a line on a screen with a camera



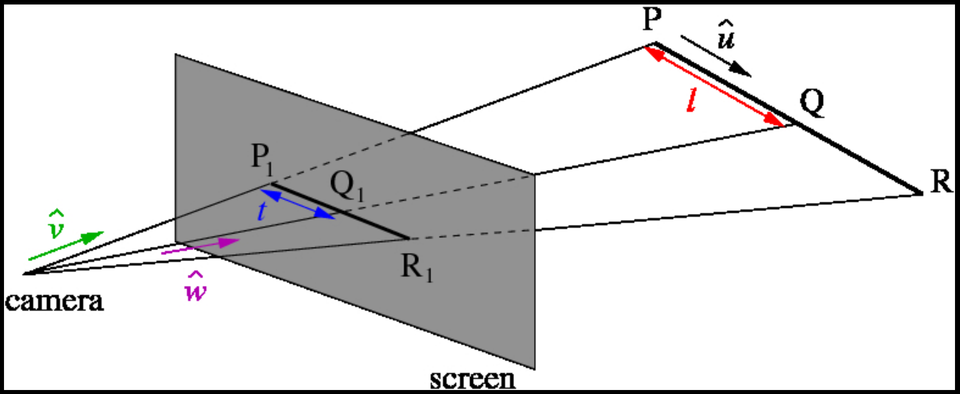
- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

  shoot a ray from the camera along $\hat{v}$, intersecting screen at P_1

  P_1: $A(x_c + av_x) + B(y_c + av_y) + C(z_c + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P_1

Projecting a line on a screen with a camera



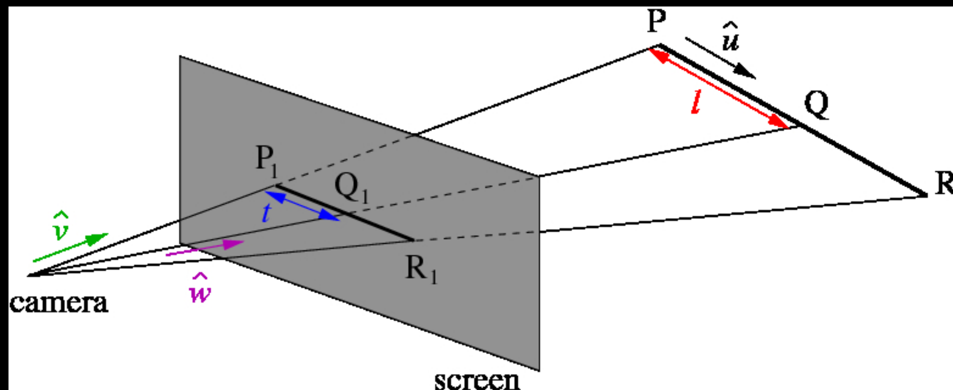
- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

  P_1: $A(x_c + av_x) + B(y_c + av_y) + C(z_c + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P_1

  determine $\hat{w}$ from camera to Q

Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera $(x_\mathrm{c}, y_\mathrm{c}, z_\mathrm{c})$, $\hat{u}$, screen $Ax+By+Cz+D=0$

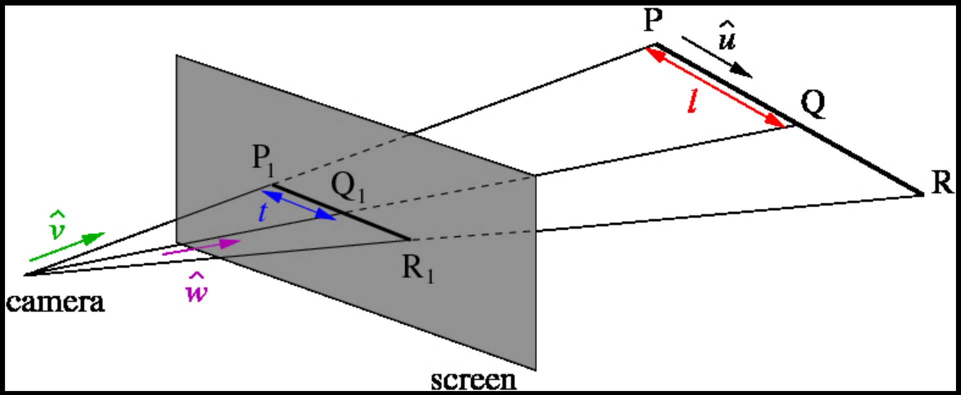
  P_1: $A(x_\mathrm{c} + av_x) + B(y_\mathrm{c} + av_y) + C(z_\mathrm{c} + av_z) + D = 0$

  $\Rightarrow$ solve a; solve for the location of P_1

  location of Q: $(x_0 + lu_x, y_0 + lu_y, z_0 + lu_z)$

  determine $\hat{w}$ from $(x_\mathrm{c}, y_\mathrm{c}, z_\mathrm{c})$ and location of Q (note: $\hat{w}$ contains l)

Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera (x_c, y_c, z_c), $\hat{u}$, screen $Ax+By+Cz+D=0$

  P_1: $A(x_c + av_x) + B(y_c + av_y) + C(z_c + av_z) + D = 0$

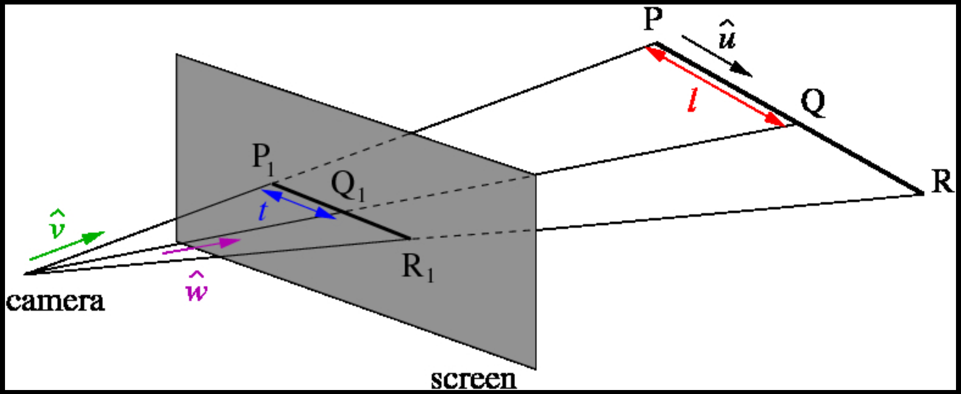
  $\Rightarrow$ solve a; solve for the location of P_1

  shoot a ray from camera to Q, intersecting screen at Q_1

  Q_1: $A(x_c + bw_x) + B(y_c + bw_y) + C(z_c + bw_z) + D = 0$

  $\Rightarrow$ solve b; solve for the location of Q_1 (note: $\hat{w}$ contains l)

Projecting a line on a screen with a camera



- Given: P (x_0, y_0, z_0), camera $(x_\mathrm{c}, y_\mathrm{c}, z_\mathrm{c})$, $\hat{u}$, screen $Ax+By+Cz+D=0$

  P_1: $A(x_\mathrm{c}+av_x)+B(y_\mathrm{c}+av_y)+C(z_\mathrm{c}+av_z)+D=0$

  $\Rightarrow$ solve a; solve for the location of P_1

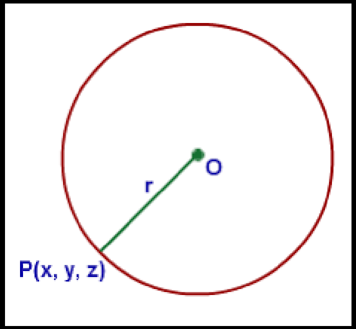
  Q_1: $A(x_\mathrm{c}+bw_x)+B(y_\mathrm{c}+bw_y)+C(z_\mathrm{c}+bw_z)+D=0$

  $\Rightarrow$ solve b; solve for the location of Q_1 (note: $\hat{w}$ contains l)

  determine t from P_1 and Q_1 (will automatically contain l)

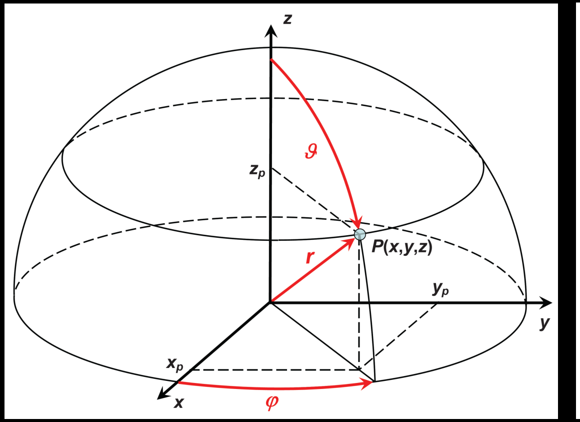
Lines and spheres in 3D

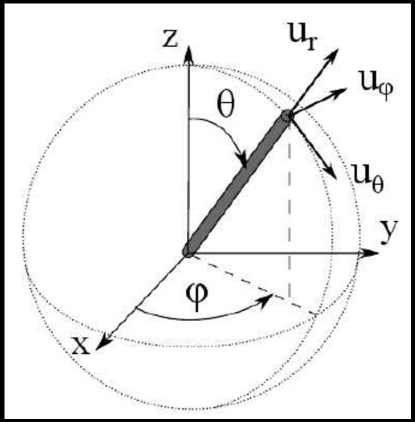
Equation of a sphere



- Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ (implicit form)
  (h,k,l): location of the centre

Spherical co-ordinate system

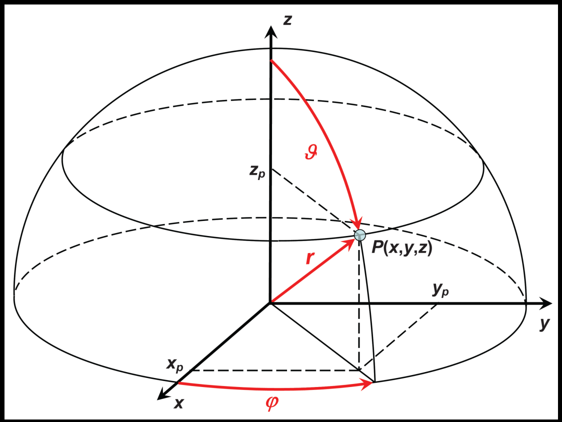


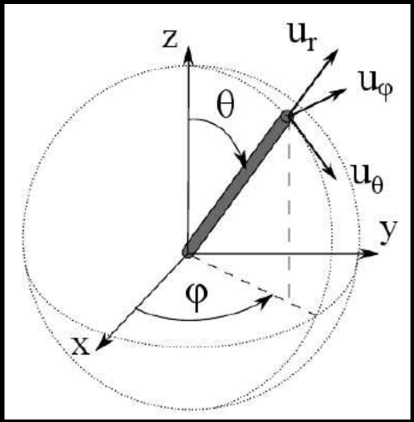


- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h \\ k \\ l \end{bmatrix} + r\,\hat{u}_r;\ \hat{u}_r = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$ (parametric form)

  $\hat{u}_r$ is the outward normal vector at P

Spherical co-ordinate system





- $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h \\ k \\ l \end{bmatrix} + r\,\hat{u}_r;\ \hat{u}_r = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$ (parametric form)

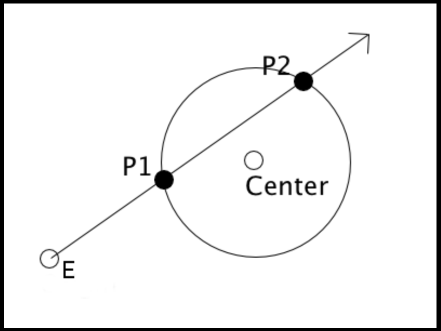
  $\hat{u}_r$ is the outward normal vector at P

  $(\hat{u}_r, \hat{u}_\theta, \hat{u}_\phi)$ for a right-handed co-ordinate system

  $\hat{u}_\theta$ and $\hat{u}_\phi$ "define" the tangent plane at P

  (any vector $\vec{v}$ on the tangent plane can be expressed as $\vec{v} = a\hat{u}_\theta + b\hat{u}_\phi$)

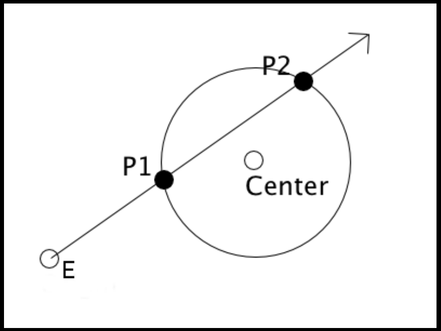
Shooting rays at a sphere



- Given: Eye at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

  Q. Find locations of P1 and P2

Shooting rays at a sphere

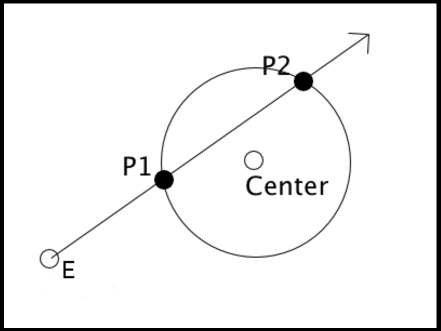


- Given: Eye at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

  Q. Find locations of P1 and P2

  A. Shoot a ray from E along $\hat{v}$, intersect sphere at P1 and P2

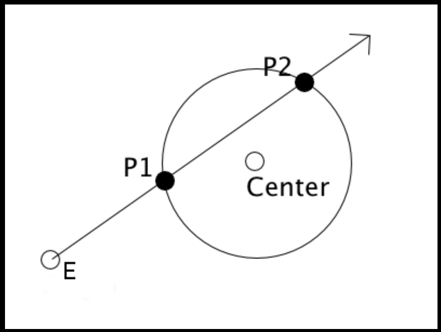
Shooting rays at a sphere



- Given: E at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}; \quad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Shooting rays at a sphere

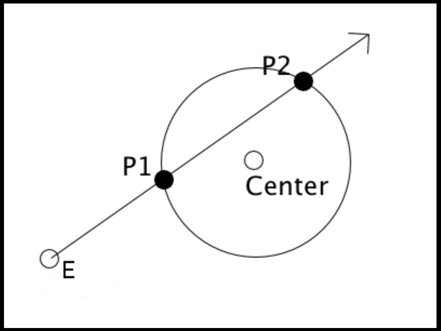


- Given: E at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}; \quad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$\Rightarrow$ quadratic equation in t: $at^2 + bt + c = 0$

Shooting rays at a sphere



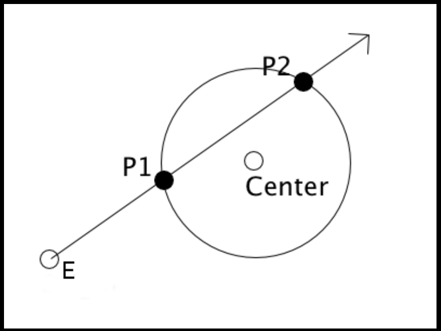
- Given: E at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}; \quad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$\Rightarrow$ quadratic equation in t: $at^2 + bt + c = 0$

$$\Rightarrow t = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Shooting rays at a sphere



- Given: E at (x_0, y_0, z_0), ray $\hat{v}$, sphere centre at (h, k, l), radius r

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}; \qquad (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

$\Rightarrow$ quadratic equation in t: $at^2 + bt + c = 0 \quad \Rightarrow \quad t = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ (i) $b^2 - 4ac > 0$: (P1,P2); (ii) $b^2 - 4ac = 0$: P1 = P2 (tangent ray); (iii) $b^2 - 4ac < 0$: no intersection; P2 will not be visible to the eye!

Summary: primitives and projections in 3D

- projections of a line on a plane
- Spheres and spherical co-ordinate system
  - surface normal and tangent planes
  - shooting rays towards a sphere and their intersections

Finally, references

- Book chapter 2: Miscellaneous Math
  - Sec. 2.2
  - Sec. 2.5, the relevant parts for 3D
- Good luck with the rest of the first half of the course!